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# Inter-species competition and chemorepulsion

J. Ignacio Tello [a,b,*], Dariusz Wrzosek [c]

[a] Depto. Matemática Aplicada a las T.I.C., ETSI Sistemas Informáticos, Universidad Politécnica de Madrid, 28031, Madrid, Spain
[b] Center for Computational Simulation, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain
[c] Inst. Applied Mathematics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland

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ABSTRACT

We study an extension of the Lotka–Volterra competition model in which one of the competing species avoids encounters with rivals by means of chemorepulsion—a repulsive reaction to the scent of competitors. The evolution of the species densities is described in terms of parabolic equations with cross-diffusive and competitive terms. The chemical signal used to detect rivals may diffuse in the environment leading to a fully parabolic problem or to a parabolic–elliptic system if the diffusion of the chemical is dominant. The most difficult case from the viewpoint of well-posedness and global existence is the parabolic–ODE system which corresponds to the case when the chemical signal does not diffuse in the environment. We prove global existence of solutions in the three cases for any space dimension for a wide range of parameters and initial conditions. For the parabolic–elliptic case, the moving rectangles method is adapted to prove the converge to a constant steady state for some range of parameters and initial data. Linear stability analysis indicates that the constant steady state in the fully parabolic case loses stability when the strength of chemorepulsion is too high and that sufficiently low degradation rate of chemorepellent stabilizes the constant steady state.


## 1. Introduction

We consider an extension of the classical model of interspecies competition due to Lotka [14] and Volterra [27] in which individuals belonging to both competing population are assumed to disperse randomly in the region which they jointly occupy. Moreover individuals of the first species try to avoid encounters with competitors by means of negative chemotaxis (chemorepulsion)—a chemo-sensory reaction to the scent of rivals. Depending on the qualitative properties of the chemical signal and the environment, the chemical (e.g. volatile odorants, allelochemicals or kairomones) may diffuse in the environment or it may be considered as

* Corresponding author at: Depto. Matemática Aplicada a las T.I.C., ETSI Sistemas Informáticos, Universidad Politécnica de Madrid, 28031, Madrid, Spain.
*E-mail addresses:* j.tello@upm.es (J.I. Tello), D.Wrzosek@mimuw.edu.pl (D. Wrzosek).

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a scent mark deposited in the environment (e.g. urinary proteins) in the case when it is not diffusive. Unlike visual or acoustic signals, olfactory signal persists in the absence of the signaler often over extended period of time. Scent marks are widely used among terrestrial vertebrates in the context of territory marking and defense (see e.g. [7]). They can be used to provide information to conspecifics or to interspecies competitors (see [5] and [30]). Denoting the densities of competing species by $u$ and $v$ and the density of the chemical signal (e.g. volatile odorant) by $w$ the model of interspecies competition with chemorepulsion reads

$$u_t = \operatorname{div}(d_u \nabla u + \chi u \nabla w) + \mu_1 u(1 - u - a_1 v), \qquad x \in \Omega,\ t > 0 \tag{1.1}$$

$$v_t = d_v \Delta v + \mu_2 v(1 - v - a_2 u), \qquad x \in \Omega,\ t > 0 \tag{1.2}$$

$$\tau w_t = d_w \Delta w - \lambda w + \alpha v, \qquad x \in \Omega,\ t > 0 \tag{1.3}$$

where $d_u > 0$, $d_v > 0$, $d_w \geq 0$, are the diffusion coefficients, $\chi > 0$ is the chemorepulsion coefficient, $\mu_1 > 0$ and $\mu_2 > 0$ are the population growth rates and $a_i > 0$ (for $i = 1, 2$) are the coefficients which describe the strength of competition. The functions $u$ and $v$ are defined on $\Omega \times (0, +\infty)$ where $\Omega \subset I\!R^N$, $N \geq 1$, is a bounded domain with smooth boundary. The system is supplemented by the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = d_w \frac{\partial w}{\partial \nu} = 0, \qquad x \in \partial\Omega \tag{1.4}$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x) \quad x \in \Omega. \tag{1.5}$$

Similarly to the Patlak–Keller–Segel model of chemotaxis (see [20] and [9]), the flux of the first species $J_u$ splits into two components, a pure diffusive flux $J_{diff}$ and a chemotactic one $J_{chem}$ to obtain

$$J_u = J_{diff} + J_{chem} = -d_u \nabla u - \chi u \nabla w.$$

The sign "$-$" in front of $\nabla w$ corresponds to a negative taxis, i.e. the movement of individuals towards decreasing concentration of the signaling chemical secreted by the individuals from the first species. The coefficients $\alpha$ and $\lambda$ in (1.3) describe the rates of signal production and signal degradation respectively. By this indirect mechanism the first species try to avoid encounters with rivals of the second species. We consider three different cases for the model:

1 Fully parabolic system ($\tau = 1$ and $d_w > 0$) when the chemical signal disperses by diffusion in the environment and satisfies a second order equation of parabolic type;
2 Parabolic–elliptic system ($\tau = 0$ and $d_w > 0$) which results from assuming the quasi-stationary approximation when the dynamics of signaling chemical is much quicker then that of species densities and the equation for the chemical is reduced to an elliptic equation;
3 Parabolic–ODE system ($\tau = 1$, $d_w = 0$) refers to the case when the chemical signal is not diffusive. The chemical signal is assumed to be deposited at spots occupied by the second species and such an olfactory cue plays the role of chemorepellent for the first species (see e.g. [7] for terrestrial ecosystems).

The first or second scenario takes place for instance in an aquatic environment (see e.g. [21], [5] and [30]) while the third in terrestial one (see e.g. [7]).

For the Lotka–Volterra model

$$\begin{aligned} u_t &= \mu_1 u(1-u-a_1 v), \\ v_t &= \mu_2 v(1-v-a_2 u), \\ u(0) = u_0 > 0\,&, v(0) = v_0 > 0 \end{aligned}$$

there are three different cases depending on the range of parameters:

- Weak competition: $0 < a_i < 1$ for $i = 1, 2$. There exists a unique globally stable steady with positive coordinates which corresponds to the coexistence of both competing species.
- Strong competition: $a_1 > 1\,, a_2 > 1$. There exists a unique steady state with positive coordinates which is unstable. Then depending on the initial data one of the two competitors stabilizes to its steady state being a winner while the other one tends to the extinction.
- Competitive exclusion of species: $(a_1 - 1)(a_2 - 1) < 0$ and $a_i > 0$ for $i = 1, 2$. There is no steady state with positive coordinates and only one competitor wins and the other one tends to the extinction ($j$-th wins if $a_i > 1\,, 0 < a_j < 1$) despite of the initial data.

The classical model does not concern inhomogeneous distribution of individuals in space and it is a natural question how the space distribution of species and their migration may affect mutual interactions of competing species. In particular it is an interesting question whether the space homogeneous steady state could lose stability if changes in space distribution due to the migration of individuals are taken into account. In such a situation spacial segregation and pattern formation are expected to occur.

A way in which the Lotka–Volterra model may be extended to incorporate spatial interactions was proposed by Shigesada, Kawasaki and Teramoto [23]. Their model takes into account the role of self-diffusion and cross-diffusion effects in the inter-species competition. Keeping the notation for $u$ and $v$ the model takes the following form

$$u_t = \mathrm{div}\{(d_u + 2a_{1,1}u + a_{12}v)\nabla u + a_{12}u\nabla v)\} + \mu_1 u(1-u-a_1 v), \tag{1.6}$$

$$v_t = \mathrm{div}\{(d_v + a_{21}u + 2a_{2,2}v)\nabla v + a_{21}v\nabla u)\} + \mu_2 v(1-v-a_2 u), \tag{1.7}$$

where $d_u\,, d_v$ are diffusion coefficients, $a_{11}, a_{22}$ are self-diffusion coefficients and $a_{12}, a_{21}$ denote cross-diffusion coefficients. The model leads to challenging mathematical problems and still some of them are unsolved. In the case of pure diffusion (i.e. $a_{i,j} = 0$ for $1 \le i,\, j \le 2$) and $\Omega$ being a convex set there is no non-constant stable steady states and in fact the constant positive steady state is a global attractor, so, in fact the dynamics is dominated by the ODE (see [3], [16] and [10]). It is worth mentioning the results of Yagi [29, Chapter 15] on the global existence of solutions to the full system (1.6)–(1.7) and the existence of global attractor for particular choice of coefficients $a_{i,j}$. For thorough discussion of population models with diffusion and advection and vast survey of related literature we refer the reader to [15], [8] or [4]. Recently Wang et al. in [28] proposed the following model which after adjusting to our notation reads

$$u_t = \mathrm{div}(d_u\nabla u + \chi u\nabla v) + \mu_1 u(1-u-a_1 v), \quad x \in \Omega,\ t > 0, \tag{1.8}$$

$$\tau v_t = d_v\Delta v + \mu_2 v(1-v-a_2 u), \quad x \in \Omega,\ t > 0\,. \tag{1.9}$$

It is easily seen that the model (1.8)–(1.9) could be considered as a special case of (1.6)–(1.7) under the assumption that only linear diffusion remains in both equations $a_{21} = a_{2,2} = a_{2,2} = 0$ and eventually only the cross-diffusion term $a_{12}u\nabla v$ remains in (1.6). On the other hand, in (1.8)–(1.9) there is a direct repulsive reaction of the first species with respect to the presence of the competitor while in our model (1.1)–(1.3), the reaction is indirect based on some chemosensory mechanism as a very scent of rivals initiates the repulsive reaction. The global existence of solutions to the system (1.8)–(1.9) for the fully parabolic case ($\tau = 1$) is proved only in the case of one space dimension. The parabolic–ODE version of (1.8)–(1.9) (i.e. $d_v = 0$) has been recently studied in [12] for weak competition terms. For space dimension $N > 1$ the solutions are global in time for the parabolic–elliptic case (i.e. $\tau = 0$). It is also shown that in the case of weak competition the constant steady state may be destabilized if we take $\chi$ large enough. Moreover the bifurcation analysis proves that for some range of parameters there are non-constant stable steady states.

The existence of global in time solutions for any space dimension for the system (1.1)–(1.3) has been studied in Section 2, Theorem 2.1. The proof is based on rather routine arguments even in fully parabolic case ($\tau = 1$).

It is worth underlining at this point that the essential difference between models is due to the boundedness of advective velocity $\nabla w$ which in our case follows easily from (1.2) and (1.3) while for system (1.8)–(1.9) an $L^\infty$-bound on $\nabla v$ is so far not known for $N > 1$. In the case of weak competition, the linear stability analysis leads to the similar condition on instability of the constant steady state as for model (1.8)–(1.9) (see Proposition 2.2). It turns out that if we keep unchanged the competition coefficients and diffusion constants in both models, the critical value of $\chi$ for which the constant steady state loses stability may be significantly increased (see Remark 2.3) if the degradation rate of the chemorepellent is small enough. In other words the appearance of a sufficiently permanent chemorepellent stabilizes the constant steady state.

In Section 3 we study the parabolic–elliptic case, global existence and stability of constant steady states is obtained by using a sub and super solution method.

Section 4 is devoted to the modification of model (1.1)–(1.3) in which chemorepellent is non-diffusive $d_w = 0$. The lack of regularity for $w$ leads to some analytical problems in the prolongation of local-in-time solutions which we overcome using essentially the dissipative structure of the reaction part of the system. It is worth noticing a simple observation that any steady state of (4.34)–(4.36) is also a steady state to system (1.8)–(1.9) with $\chi$ replaced by $\tilde{\chi} := \frac{\lambda}{\alpha}\chi$ and vice versa. Bifurcation analysis performed in [28] for system (1.8)–(1.9) implies that for some range of parameters our system (1.1)–(1.3) also possesses non-constant steady states. In Theorem 4.2 it is proved that the solution to the model with non-diffusive chemorepellent is global in time and uniformly bounded for any space dimension.

## 2. The fully parabolic system: global existence of solutions and linear stability

**Theorem 2.1.** *Suppose that $u_0, v_0, w_0 \in W^{1,p}(\Omega)$, $p > N$ are nonnegative functions. Then, for $\tau = 1$ and $d_w > 0$ there exists a unique global classical solution to (1.1)–(1.3) defined on $\Omega \times (0, +\infty)$.*

**Proof.** In the case $\tau = 1$ we may notice that the main part of the quasilinear parabolic system is a normally elliptic operator with upper-triangular structure and the existence and uniqueness of maximal classical solution defined on $\Omega \times (0, T_{max})$ follows from Amann's theory [2, Theorems 14.4 & 14.6]. The nonnegativity of solutions easily follows from the maximal principle. Since there is $K > 0$ such that $u(1 - u - a_1 v) \le K - u$ for $u, v \ge 0$ integrating the first equation on $\Omega$ and using (1.4) we easily infer that for some constant $C > 0$

$$\int_\Omega u(x,t)dx \le C \quad \text{for} \quad t \in [0, T_{max}), \tag{2.10}$$

and by the comparison technique with logistic ODE applied to (1.2) we obtain that $v$ is an $L^\infty$-bounded function and

$$0 \le v(\cdot, t) \le \max\{1, (1 + (\|v_0\|_\infty^{-1} - 1)e^{-\mu_2 t})^{-1}\} \quad \text{for} \quad t \in [0, T_{max}). \tag{2.11}$$

From [11] and (1.3) we obtain (2.11) and thanks to parabolic regularity theory we deduce that there exists a constant $C_2$ such that

$$\|w(\cdot\,,t)\|_{W^{1,\infty}(\Omega)} \le C_2 \quad \text{for} \quad t \in [0\,,T_{max}). \tag{2.12}$$

Applying the well-known Alikakos–Moser technique [1] to (1.1), essentially in the same way as [28], we obtain that there is a constant $C_3$ independent of time such that

$$\|u(\cdot,t)\|_\infty \le C_3 \quad \text{for} \quad t \in [0\,,T_{max}). \tag{2.13}$$

Finally, it follows from (2.11)–(2.13) that

$$\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty + \|w(\cdot,t)\|_\infty \le C_5 \quad \text{for} \quad t \in [0\,,T_{max}), \tag{2.14}$$

and due to the fact that the main part of the operator is upper-triangular, it follows from [2, Theorem 15.5] that the uniform $L^\infty$ bound (2.14) ensures the extendability criterion for the maximal solution to get $T_{max} = +\infty$. $\square$

### *2.1. Steady states and linearization*

In this section we study the stability of the positive constant steady state

$$P = (u^*\,,v^*\,,w^*) = \left(\frac{1-a_1}{1-a_1a_2}\,,\frac{1-a_2}{1-a_1a_2},\frac{\alpha(1-a_2)}{\lambda(1-a_1a_2)}\right) \tag{2.15}$$

under assumption of weak competition, i.e.

$$0 < a_i < 1, \qquad \text{for } i = 1,2 \tag{2.16}$$

which corresponds to the stable coexistence of both species in the frame of the ODE model. We shall show that chemorepulsion may destabilize such a steady state.

Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots\,, \tag{2.17}$$

of operator $-\Delta$ defined on the space

$$W_B^{2,p} = \left\{\varphi \in W^{2,p}(\Omega) : \frac{\partial\psi}{\partial\nu} = 0\right\}.$$

Notice also that the linear part of the operator in (1.1)–(1.3) is normally elliptic and it has only point spectrum. Standard linearization (see e.g. [13, Lemma 2.1]) of the system (1.1)–(1.3) at the steady state $P$ leads to the following matrix

$$D_n = \begin{bmatrix} -d_u\lambda_n - \mu_1 u^* & -\mu_1 a_1 u^* & -\chi u^*\lambda_n \\ -\mu_2 a_2 v^* & -(d_v\lambda_n + \mu_2 v^*) & 0 \\ 0 & \alpha & -d_w\lambda_n - \lambda \end{bmatrix}. \tag{2.18}$$

**Proposition 2.2.** *Under assumption (2.16), $P$, the steady state of (1.1)–(1.3) is locally asymptotically stable if*

$$\chi < \chi_0 := \min_{n \geq 1} \frac{(d\lambda_n + \Psi)\,(E_n + (d_w\lambda_n + \lambda)((d + d_w)\lambda_n + \Psi + \lambda))}{u^* v^* \mu_2 a_2 \alpha \lambda_n} \tag{2.19}$$

*where*

$$d = d_u + d_v\,, \quad \Psi = \mu_1 u^* + \mu_2 v^* \tag{2.20}$$

*and*

$$E_n = (d_u\lambda_n + \mu_1 u^*)(d_v\lambda_n + \mu_2 v^*) - \mu_1\mu_2 a_1 a_2 u^* v^* > 0\,. \tag{2.21}$$

*For $\chi > \chi_0$ large enough, the steady state "$P$" is unstable.*

**Proof.** The steady state $P$ is locally stable if real parts of eigenvalues of matrix $D_n$ are negative for $n \geq 1$, the case of $D_0$ will be considered separately. $D_n$ has the following characteristic polynomial

$$q(\sigma) = \sigma^3 + A_2(n)\sigma^2 + A_1(n)\sigma + A_0(n)$$

where

$$\begin{aligned} A_2(n) &= (d_u + d_v + d_w)\lambda_n + \Psi + \lambda\,, \\ A_1(n) &= E_n + (d_w\lambda_n + \lambda)[d\lambda_n + \Psi]\,, \\ A_0(n) &= \chi u^* v^* \mu_2 a_2 \alpha \lambda_n + (d_w\lambda_n + \lambda)E_n\,, \end{aligned}$$

with $d\,, \Psi\,, E_n$ being defined in (2.20)–(2.21). Notice that

$$A_0(n) > 0\,, A_1(n) > 0 \quad \text{for} \;\; n \geq 1$$

and it follows from the Routh–Hurwitz criterion (see e.g. Murray [17], Appendix B.1) that all eigenvalues of $D_n$ have negative real parts if and only if for all $n \geq 1$

$$A_0(n) < A_1(n)A_2(n). \tag{2.22}$$

After lengthy but straightforward calculations (2.22) turns out to be equivalent to

$$p(\lambda_n) := (d\lambda_n + \Psi)(E_n + (d_w\lambda_n + \lambda)((d + d_w)\lambda_n + \Psi + \lambda)) > \chi u^* v^* \mu_2 a_2 \alpha \lambda_n\,.$$

Notice that $p(x)$ is a polynomial of the third order with positive coefficients and $p(0) > 0$. Now it is clear that for $x > 0$ the function $x \longmapsto \frac{p(x)}{u^* v^* \mu_2 a_2 \alpha x}$ has a positive minimum and there exist

$$\chi_0 = \min_{n \geq 1} \frac{p(\lambda_n)}{u^* v^* \mu_2 a_2 \alpha \lambda_n}\,.$$

Hence the stability condition follows for $\chi < \chi_0$. Notice that for the case $n = 0$ we have $\lambda_0 = 0\,,\; E_0 = \mu_1\mu_2 u^* v^*(1 - a_1 a_2) > 0$ and it is easy to check that $A_i(0) > 0$ for all $i$ and (2.22) holds which implies that all the eigenvalues of $D_0$ are negative.

Using again the Routh–Hurwitz criterion we infer that $P$ is unstable if for some $\chi > \chi_0$ there exists $\lambda_n$ such that

$$p(\lambda_n) < \chi u^* v^* \mu_2 a_2 \alpha \lambda_n \,.$$

The last observation completes the proof. $\square$

**Remark 2.3.** The stability condition of the state $(u^*, v^*)$ for model (1.8)–(1.9) from [28] is the following

$$\chi < \chi_1 := \min_{n\geq 1} \frac{E_n}{u^* v^* \mu_2 a_2 \lambda_n} \tag{2.23}$$

where the original notation used in [28] is suitably adjusted. Notice that $\chi_0 > \chi_1$ provided $\frac{(d\lambda_1+\Psi)}{\alpha} > 1$ whence we conclude that the range of stability for parameter $\chi$ in our model is wider than that in model (1.8)–(1.9) provided the degradation rate of chemorepellent is small enough.

If some value of $\chi = \chi'$ is in the instability region for model (1.8)–(1.9) and we keep unchanged the competition and diffusion coefficients in both models then the degradation coefficient of chemorepellent may be taken small enough such that $\chi'$ is still in the stability region of the constant steady state in our model.

## 3. Parabolic–elliptic case

In this section we study the long time behavior of solutions to the parabolic–elliptic model when $\tau = 0$, then

$$u_t = \operatorname{div}(d_u \nabla u + \chi u \nabla w) + \mu_1 u(1 - u - a_1 v), \; x \in \Omega, \; t > 0 \tag{3.24}$$

$$v_t = d_v \Delta v + \mu_2 v(1 - v - a_2 u), \qquad x \in \Omega, \; t > 0 \tag{3.25}$$

$$0 = d_w \Delta w - \lambda w + \alpha v, \qquad x \in \Omega, \; t > 0. \tag{3.26}$$

**Theorem 3.1.** *Suppose that $u_0, v_0 \in W^{1,p}(\Omega)$, $p > N$ are nonnegative functions. Then, for $\tau = 0$ and $d_w > 0$ there exists a unique global classical solution to (1.1)–(1.3) defined on $\Omega \times (0, \infty)$.*

**Proof.** To obtain the global existence of solutions, we split the non-linear second order term in the following way

$$\operatorname{div}(\chi u \nabla w) = \chi \nabla u \nabla w + \chi u \Delta w = \chi \nabla u \nabla w + \frac{\chi}{d_w} u(\lambda w - \alpha v)$$

and replace it in the equation. We now proceed as in the Theorem 2.1 to obtain that $u \geq 0$ and the uniform bounds for $\|v\|_{L^\infty(\Omega)}$ and $\|w\|_{W^{1,\infty}(\Omega)}$. The rest of the proof follows the arguments of [25, Lemma 2.1]. $\square$

The following results shows that for some range of parameters the basin of attraction of steady state $P$ defined in (2.15) may be estimated.

**Theorem 3.2.** *Let $0 < a_i < 1$ for $i = 1, 2$, then, under assumption*

$$\frac{\mu_1}{a_2} > \frac{2\chi\alpha}{d_w} + \mu_1 a_1, \quad \mu_2 > 0 \tag{3.27}$$

*for bounded initial data $u_0$, $v_0$ satisfying*

$$0 < \min_{x\in\overline{\Omega}}\{u_0\} \le \max_{x\in\overline{\Omega}}\{u_0\} < \infty$$

$$0 < \min_{x\in\overline{\Omega}}\{v_0\} \le \max_{x\in\overline{\Omega}}\{v_0\} < \infty,$$

*the solution to the problem (3.24)–(3.26) satisfies*

$$\|u(t)-u^*\|_{L^\infty(\Omega)} + \|v(t)-v^*\|_{L^\infty(\Omega)} + \|w(t)-w^*\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t\to\infty.$$

**Proof.** The nonlinear second order term is treated as follows:

$$\operatorname{div}(\chi u\nabla w) = \chi\nabla u\nabla w + \chi u\Delta w = \chi\nabla u\nabla w + \frac{\chi}{d_w}u(\lambda w - \alpha v)$$

then, the equation (3.24) reads

$$u_t = d_u\Delta u + \chi\nabla u\nabla w + \frac{\chi}{d_w}u(\lambda w-\alpha v) + \mu_1 u(1-u-a_1 v),\ x\in\Omega,\ t>0$$

i.e.

$$u_t = d_u\Delta u + \chi\nabla u\nabla w + \mu_1 u\left[1+\frac{\chi\lambda}{d_w\mu_1}w - u - \left(a_1+\frac{\chi\alpha}{d_w\mu_1}\right)v\right],\ x\in\Omega,\ t>0.$$

We consider the following system of ODEs

$$\overline{u}_t = \mu_1\overline{u}\left[1+\frac{\chi\alpha}{d_w\mu_1}\overline{v} - \overline{u} - \left(a_1+\frac{\chi\alpha}{d_w\mu_1}\right)\underline{v}\right] \tag{3.28}$$

$$\underline{u}_t = \mu_1\underline{u}\left[1+\frac{\chi\alpha}{d_w\mu_1}\underline{v} - \underline{u} - \left(a_1+\frac{\chi\alpha}{d_w\mu_1}\right)\overline{v}\right] \tag{3.29}$$

$$\overline{v}_t = \mu_2\overline{v}(1-\overline{v}-a_2\underline{u}), \tag{3.30}$$

$$\underline{v}_t = \mu_2\underline{v}(1-\underline{v}-a_2\overline{u}), \tag{3.31}$$

satisfying the following initial conditions

$$\overline{u}(0)=\overline{u}_0,\quad \underline{u}(0)=\underline{u}_0,\quad \overline{v}(0)=\overline{v}_0,\quad \underline{v}(0)=\underline{v}_0 \tag{3.32}$$

under assumption

$$0<\underline{u}_0<u^*<\overline{u}_0,\quad 0<\underline{v}_0<v^*<\overline{v}_0. \tag{3.33}$$

Notice that, standard ODE theory gives the global existence, positivity and uniformly boundedness of solutions of the system (3.28)–(3.31). The next three lemmatta will lead to completion of the proof of the theorem

**Lemma 3.3.** *Under assumptions (3.27) and (3.33) the solution to the ODE system (3.28)–(3.31) satisfies the following order relation*

$$\underline{u}(t)<\overline{u}(t),\quad \underline{v}(t)<\overline{v}(t)\qquad \text{for } t>0.$$

**Proof.** We argue by contradiction and assume that there exists $t_0 > 0$ such that

$$(\overline{u}(t_0) - \underline{u}(t_0))(\overline{v}(t_0) - \underline{v}(t_0)) = 0$$

and

$$(\overline{u}(t) - \underline{u}(t))(\overline{v}(t) - \underline{v}(t)) > 0, \qquad \text{for } t < t_0.$$

Then, three situations may occur:

Case 1. $\overline{u}(t_0) = \underline{u}(t_0)$ and $\overline{v}(t_0) > \underline{v}(t_0)$;
Case 2. $\overline{u}(t_0) > \underline{u}(t_0)$ and $\overline{v}(t_0) = \underline{v}(t_0)$;
Case 3. $\overline{u}(t_0) = \underline{u}(t_0)$ and $\overline{v}(t_0) = \underline{v}(t_0)$.

The positivity of the solutions gives that in Case 1,

$$\overline{u}_t(t) > \overline{u}_t(t), \qquad \text{for } t \leq t_0$$

which contradicts $\overline{u}(t_0) = \underline{u}(t_0)$ and (3.33). In the same way Case 2 contradicts $\overline{v}(t_0) = \underline{v}(t_0)$ and (3.33). In Case 3, we consider the system

$$\begin{cases} \tilde{u}_t = \mu_1 \tilde{u} \left[1 + \frac{\chi\alpha}{d_w \mu_1} \tilde{v} - \tilde{u} - \left(a_1 + \frac{\chi\alpha}{d_w \mu_1}\right) \tilde{v}\right] \\ \tilde{v}_t = \mu_2 \tilde{v}(1 - \tilde{v} - a_2 \tilde{u}), \\ \tilde{u}(t_0) = \underline{u}(t_0) = \overline{u}(t_0), \qquad \tilde{v}(t_0) = \underline{v}(t_0) = \overline{v}(t_0), \end{cases}$$

which has a unique solution in $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$ small enough. Notice that

$$\underline{u} = \overline{u} = \tilde{u}, \qquad \underline{v} = \overline{v} = \tilde{v},$$

is a solution to the ODE system (3.28)–(3.31). Uniqueness of solutions of (3.28)–(3.31) guarantee that

$$\overline{u}(t) = \underline{u}(t), \quad \overline{v}(t) = \underline{v}(t), \text{ for any } t \in (t_0 - \delta, t_0 + \delta)$$

which contradicts the definition of $t_0$ and ends the proof. $\square$

**Lemma 3.4.** *Under assumptions (3.27) and (3.33) the solution to the ODE system (3.28)–(3.31) satisfies*

$$|\underline{u}(t) - \overline{u}(t)| + |\underline{v}(t) - \overline{v}(t)| \to 0 \qquad \text{as } t \to \infty.$$

**Proof.** We now divide (3.28) by $\overline{u}$, (3.29) by $\underline{u}$, (3.30) by $\overline{v}$ and (3.31) by $\underline{v}$ to obtain,

$$\frac{\overline{u}_t}{\overline{u}} = \mu_1 \left[1 + \frac{\chi\alpha}{d_w \mu_1} \overline{v} - \overline{u} - \left(a_1 + \frac{\chi\alpha}{d_w \mu_1}\right) \underline{v}\right]$$
$$\frac{\underline{u}_t}{\underline{u}} = \mu_1 \left[1 + \frac{\chi\alpha}{d_w \mu_1} \underline{v} - \underline{u} - \left(a_1 + \frac{\chi\alpha}{d_w \mu_1}\right) \overline{v}\right]$$
$$\frac{\overline{v}_t}{\overline{v}} = \mu_2(1 - \overline{v} - a_2 \underline{u})$$
$$\frac{\underline{v}_t}{\underline{v}} = \mu_2(1 - \underline{v} - a_2 \overline{u}).$$

We combine the equations for $\underline{u}$ and $\overline{u}$ to get

$$\frac{d}{dt}(\log \overline{u} - \log \underline{u}) = (\frac{2\chi\alpha}{d_w} + \mu_1 a_1)(\overline{v} - \underline{v}) - \mu_1(\overline{u} - \underline{u})$$
$$\frac{d}{dt}(\log \overline{v} - \log \underline{v}) = \mu_2 a_2(\overline{u} - \underline{u}) - \mu_2(\overline{v} - \underline{v}).$$

We multiply the last equation by a constant $A$ satisfying

$$A\mu_2 a_2 < \mu_1, \qquad A\mu_2 > \frac{2\chi\alpha}{d_w} + \mu_1 a_1.$$

Notice that such $A$ exists in view of assumption (3.27), then

$$\frac{d}{dt}\left(\log\frac{\overline{u}}{\underline{u}} + A\log\frac{\overline{v}}{\underline{v}}\right) \le -\epsilon(\overline{u} - \underline{u} + \overline{v} - \underline{v})$$

for

$$\epsilon < \min\left\{A\mu_2 - \left(\frac{2\chi}{d_w} + \mu_1 a_1\right), \mu_1 - A\mu_2 a_2\right\}.$$

We now proceed as in [24, Theorem 5.1] to deduce that

$$|\underline{u}(t) - \overline{u}(t)| + |\underline{v}(t) - \overline{v}(t)| \to 0 \quad \text{as } t \to \infty$$

and the proof ends. $\square$

**Lemma 3.5.** *Under assumptions of Theorem 3.2, the solution $(u, v, w)$ to (3.24)–(3.26) satisfies*

$$\underline{u}(t) \le u(t,x) \le \overline{u}(t), \quad \underline{v}(t) \le v(t,x) \le \overline{v}(t,x), \quad \frac{\alpha}{\lambda}\underline{v}(t) \le w(t,x) \le \frac{\alpha}{\lambda}\overline{v}(t,x)$$

*for* $(t,x) \in [0,\infty) \times \Omega$.

**Proof.** The proof is similar to the proof of [18, Theorem 2.1], therefore we only present the main idea of the proof:

We consider $T < \infty$ and the following functions

$$\overline{U}(x,t) := u(x,t) - \overline{u}(t), \quad \underline{U}(x,t) := u(x,t) - \underline{u}(t),$$
$$\overline{V}(x,t) := v(x,t) - \overline{v}(t), \quad \underline{V}(x,t) := v(x,t) - \underline{v}(t),$$
$$\overline{W}(x,t) := w(x,t) - \frac{\alpha}{\lambda}\overline{v}(t), \quad \underline{W}(x,t) := u(x,t) - \frac{\alpha}{\lambda}\underline{v}(t),$$

which satisfy the system of equations

$$\overline{U}_t - d_u\Delta\overline{U} = \chi\nabla\overline{U}\nabla\overline{W} + g_{11}\overline{U} + g_{12}\underline{V} + g_{13}\overline{W}, \qquad x\in\Omega,\ t\in(0,T)$$
$$\underline{U}_t - d_u\Delta\underline{U} = \chi\nabla\underline{U}\nabla\underline{W} + g_{21}\underline{U} + g_{22}\overline{V} + g_{23}\underline{W}, \qquad x\in\Omega,\ t\in(0,T)$$
$$\overline{V}_t - d_v\Delta\overline{V} = g_{31}\overline{V} + g_{32}\underline{U}, \qquad x\in\Omega,\ t>0$$
$$\underline{V}_t - d_v\Delta\underline{V} = g_{41}\underline{V} + g_{42}\overline{U}, \qquad x\in\Omega,\ t>0$$
$$-d_w\Delta\overline{W} + \lambda\overline{W} = \alpha\overline{V}, \qquad x\in\Omega,$$
$$-d_w\Delta\underline{W} + \lambda\underline{W} = \alpha\underline{V}, \qquad x\in\Omega,$$

where $g_{ij} = g_{ij}(u, v, w, \overline{u}, \underline{u}, \overline{v}, \underline{v})$ (for $i = 1 \dots 4$ and $j = 1 \dots 3$) are continuous functions and

$$g_{12} \leq 0, \quad g_{13} \geq 0, \quad g_{22} \leq 0, \quad g_{23} \geq 0, \quad g_{32} \leq 0 \text{ and } g_{42} \leq 0.$$

We multiply the previous equations by $\overline{U}_+$, $\underline{U}_-$, $\overline{V}_+$, $\underline{V}_-$, $\overline{W}_+$, $\underline{W}_-$ respectively and after routine computations we obtain

$$\frac{d}{dt}\int_\Omega \overline{U}_+^2 + \underline{U}_-^2 + \overline{V}_+^2 + \underline{V}_-^2 \leq K(T)\int_\Omega \overline{U}_+^2 + \overline{U}_-^2 + \overline{V}_+^2 + \overline{V}_-^2, \qquad \text{for } t < T.$$

We apply Gronwall's inequality to the previous inequality to obtain

$$\overline{U}_+ = \underline{U}_- = \overline{V}_+ = \overline{V}_- = 0, \qquad \text{for } \ t < T$$

and thanks to maximum principle we also get

$$\overline{W}_+ = \overline{W}_- = 0.$$

To end the proof we take limits as $T \to \infty$. $\square$

**End of the proof of the Theorem 3.2.** Now, the proof of the Theorem follows from Lemma 3.4 and 3.5. $\square$

## 4. Model with non-diffusive chemorepellent

In this section we study the case of non-diffusing signaling chemical i.e. $d_w = 0$ which in this case may be interpreted as an olfactory cue left by rivals from the other population. Then the model reads

$$u_t = \operatorname{div}(d_u \nabla u + \chi u \nabla w) + \mu_1 u(1 - u - a_1 v), \quad x \in \Omega, \ t > 0, \tag{4.34}$$

$$v_t = d_v \Delta v + \mu_2 v(1 - v - a_2 u), \quad x \in \Omega, \ t > 0, \tag{4.35}$$

$$\tau w_t = -\lambda w + \alpha v, \quad x \in \Omega, \ t > 0. \tag{4.36}$$

Notice that there is a correspondence between the steady states of (4.34)–(4.36) and those of (1.8)–(1.9). Namely each steady state $P = (\bar{u}, \bar{v}, \bar{w})$ of (4.34)–(4.36) satisfies $\bar{v} = \frac{\lambda}{\alpha}\bar{w}$ which implies that $(\bar{u}, \bar{v})$ is a steady state of (1.8)–(1.9) with $\chi$ replaced by $\tilde{\chi} := \frac{\lambda}{\alpha}\chi$. It is easily seen that the inverse statement also holds true. The following statement is an immediate consequence of [28, Theorem 3.1].

**Proposition 4.1.** *For space dimension $N = 1$ and some $\chi > \frac{\lambda}{\alpha}\chi_1$, where $\chi_1$ was defined in (2.23), the system (4.34)–(4.36) possesses non-constant steady states.*

To analyze the time dependent problem we first introduce the following change of unknown functions, from $u$ to $z$

$$z := u e^{\frac{\chi}{d_u} w} \tag{4.37}$$

being a usual trick which simplifies analysis of chemotaxis models with non-diffusing chemoattractant (this is the case of models of tumor progression which take into account haptotaxis). Then on substituting to (4.34) we find

$$u_t = e^{-\frac{\chi}{d_u} w}(z_t - z\frac{\chi}{d_u} w_t) = e^{-\frac{\chi}{d_u} w}(z_t - \frac{\chi}{d_u \tau} z(\alpha v - \lambda w))$$

and

$$\operatorname{div}(d_u \nabla u + \chi u \nabla w) = e^{-\frac{\chi}{d_u} w}(d_u \Delta z - \chi \nabla z \nabla w)\,.$$

Then multiplying (4.34) by $e^{-\frac{\chi}{d_u} w}$ we arrive at

$$z_t = d_u \Delta z - \chi \nabla z \nabla w + \mu_1 z(1 - z e^{-\frac{\chi}{d_u} w} - a_1 v) + \frac{\chi}{d_u \tau} z(\alpha v - \lambda w), \tag{4.38}$$

$$v_t = d_v \Delta v + \mu_2 v(1 - v - a_2 z e^{-\frac{\chi}{d_u} w}), \tag{4.39}$$

$$\tau w_t = -\lambda w + \alpha v, \tag{4.40}$$

defined on $\Omega \times (0, \infty)$ with homogeneous Neumann boundary conditions on $z$ and $v$. By denoting

$$Z(t) = (z(t)\,, v(t))^T \quad \text{and} \quad F = (F_1\,, F_2)^T$$

and using the variation of constant formula the system may be transformed into the form

$$Z(t) = Z(0) + \int_0^t e^{-(t-s)A} F(Z(s)\,, w(s))ds \tag{4.41}$$

$$w(t) = w_0 e^{-\lambda t} + \alpha \int_0^t e^{-\lambda(t-s)} v(s)ds \tag{4.42}$$

where $A = (A_u\,, A_v)^T = (-d_u \Delta + Id\,, -d_v \Delta + Id)^T$ is defined on $W_B^{2,p} \times W_B^{2,p}$ for

$$W_B^{s,p} = W_B^{s,p}(\Omega) = \{\varphi \in W^{s,p}(\Omega) : \frac{\partial \psi}{\partial \nu} = 0\}$$

and

$$F_1(Z\,, w) = z - \chi \nabla z \nabla w + \mu_1 z(1 - z e^{-\frac{\chi}{d_u} w} - a_1 v) + \frac{\chi}{d_u \tau} z(\alpha v - \lambda w)\,,$$

$$F_2(Z\,, w) = v + \mu_2 v(1 - v - a_2 z e^{-\frac{\chi}{d_u} w})\,.$$

Notice that $A$ is a generator of an analytic semigroup in $L^p(\Omega) \times L^p(\Omega)$ and fractional power space $X_p^\gamma = D(A_u^\gamma) = D(A_v^\gamma)$ is well defined for $\gamma \in [0\,, 1]$ (see. e.g. [6, Definition 1.4.7]). We fix $\gamma$ such that $1 > \gamma > \frac{1}{2} + \frac{N}{2p}$. Using again [6, Theorem 1.6.1] it entails that

$$X_p^\gamma \subset C^1(\bar{\Omega})\,. \tag{4.43}$$

It follows from [29, Theorem 16.11] that for $1 > \gamma > \frac{1}{2} + \frac{N}{2p}$

$$X_p^\gamma = W_B^{2\gamma,p}(\Omega)\,,$$

where $W^{2\gamma,p}(\Omega)$ is the fractional order Sobolev space.

The first result of this section is the existence of global solutions, the result is enclosed in the following theorem.

**Theorem 4.2.** *Suppose that $p > N$, the initial functions are nonnegative and $u_0\,, v_0\, w_0 \in W_B^{\gamma',p}(\Omega)$ for some $2 > \gamma' > 1 + \frac{N}{p}$. Then there exists a unique global-in-time solution $(u, v, w)$ to (4.34)–(4.36) such that for any $T > 0$*

$$u\,, v\,, w \in C([0, T] : W_B^{\gamma',p}(\Omega))$$

*with uniform in time $L^\infty$-bound. Moreover $u\,, v\,, w \in C^{2,1}_{x,t}(\bar\Omega \times (0\,, T))$.*

The proof of Theorem 4.2 is preceded by two propositions. The following statement concerns the existence of local-in-time solution to the auxiliary problem (4.38)–(4.40).

**Proposition 4.3.** *For any $N \geq 1$ and $z_0\,, v_0\,, w_0 \in X_p^\gamma$ there exists $T_{max}$ such that $z\,, v\,, w \in C([0, T_{max}) : X_p^\gamma)$ is a unique solution to (4.38)–(4.40) and*

$$T_{max} = \infty \quad or \;\; T_{max} < \infty \;\; and \;\; \lim_{t\to T_{max}} \max\{\|z(t)\|_{X_p^\gamma}, \|v(t)\|_{X_p^\gamma}\} = \infty\,.$$

*Moreover $z\,, v\,, w \in C^{2,1}_{x,t}(\bar\Omega \times (0\, T_{max}))$.*

**Proof.** The existence and uniqueness of local-in-time mild solution $z\,, v\,, w \in C([0, T_{max}) : X_p^\gamma)$ satisfying (4.41)–(4.42) (see e.g. [6] or [22]) is proved by means of the Banach Contraction theorem using routine arguments based on the fact that due to (4.43) $F$ in (4.41) is a locally Lipschitz continuos function in the following sense:
For any $(Z_1, w_1)^T\,, (Z_2\,, w_2)^T \in (X_p^\gamma)^3)$ such that $\|(Z_i, w_i)^T\|_{(X_p^\gamma)^3} \leq R$ there exists a constant $C(R)$ such that

$$\|F(Z_1, w_1) - F(Z_2, w_2)\|_{L^p(\Omega)} \leq C(R)\|Z_1 - Z_2\|_{(X_p^\gamma)^2} + \|w_1 - w_2\|_{X_p^\gamma}\,. \tag{4.44}$$

In order to raise the regularity of the mild solution notice that for fixed $w$ given by (4.42) and any $T < T_{max}$ for any $0 \leq t_1 < t_2 \leq T$ we have

$$\|w(t_2) - w(t_1)\|_{X_p^\gamma} \leq \|w_0\|_{X_p^\gamma}\lambda|t_2 - t_1| + T\alpha\lambda \sup_{t\in[0,T]} \|v(t)\|_{X_p^\gamma}|t_2 - t_1| \tag{4.45}$$

and hence we may define $\tilde F(Z, t) = F(Z\,, w(t))$ and the first two equations may be considered in the abstract form

$$Z_t + AZ = \tilde F(Z\,, t) \quad Z(0) = Z_0.$$

It follows from (4.44) and (4.45) that there exists a constant $C'$ such that

$$\|\tilde F(Z_2\,, t_2) - \tilde F(Z_1\,, t_1)\|_{L^p(\Omega)} \leq C'(\|Z_1 - Z_2\|_{(X_p^\gamma)^2} + |t_2 - t_1|)$$

Classical results from [6] entails that

$$z, v \in C(0\,, T; W_B^{2,p})\,, z_t\,, v_t \in C(0\,, T; (L^p(\Omega))) \tag{4.46}$$

and in view of (4.42) we easily deduce that

$$w \in C^1(0\,, T; (W_B^{2,p})) \tag{4.47}$$

The parabolic regularity theory implies that $u\,, v\,, w \in C^{2,1}_{x,t}(\bar{\Omega} \times (0\,T_{max}))$. The uniqueness of solution is an immediate consequence of the regularity of solutions and local Lipshitz continuity of the nonlinear terms. $\square$

Let us notice that thanks to the regularity of functions, the solution $(z, v, w)$ to (4.38)–(4.39) defines the solution to our original problem $(u, v, w)$ to (4.34)–(4.36). We proceed to prove that the maximal solution is uniformly bounded in $L^\infty$. Observe that (2.11) implies that there is a constant $v_M$ independent on time such that

$$v_M := \sup_{t\in[0\,,T_{max}]} \|v(t)\|_\infty$$

and (4.36) implies that

$$\sup_{t\in[0\,,T_{max})} \|w(t)\|_\infty \le \max\{\|w_0\|_\infty, \frac{\alpha v_M}{\lambda}\} := w_M$$

it remains to show that there is a constant $u_M$

$$u_M := \sup_{t\in[0\,,T_{max})} \|u(t)\|_\infty < \infty\,.$$

The proof of $L^\infty$ boundedness is presented in the following two lemmas. The main idea is adapted from [19, Lemma 2.5] and it is based on introducing an iteration with respect to $p$ for the norm $L^\infty(0, T_{max} : L^p(\Omega))$.

**Remark 4.4.** Since the initial data satisfy $u_0 \ge 0$, $v_0 \ge 0$ and $w_0 \ge 0$ we obtain

$$u \ge 0, \quad v \ge 0, \quad \text{and} \quad w \ge 0 \quad \text{for} \quad t \in [0, T_{max}). \tag{4.48}$$

Let us denote

$$f(w) = e^{\frac{\chi}{d_u} w}$$

and notice that

$$f'(w) = \frac{\chi}{d_u} f(w).$$

**Lemma 4.5.** *Let $(u, v, w)$ be a solution to (4.34)–(4.36). For $p > 1$ and $t \in (0\,, T_{max})$ it holds*

$$\frac{1}{p-1}\frac{d}{dt}\int_\Omega u^p f^{p-1} dx \le -a \int_\Omega u^{p+1} f^p dx + b \int_\Omega u^p f^{p-1} dx, \tag{4.49}$$

*for*

$$a := \frac{p\mu_1}{(p-1)e^{\frac{\chi}{d_u} w_M}} \qquad \text{and} \qquad b := \frac{p\mu_1}{p-1} + \frac{\chi}{d_u \tau}\alpha v_M.$$

**Proof.** We first compute

$$\frac{d}{dt}\int_\Omega u^p f^{p-1} dx = p\int_\Omega u^{p-1} u_t f^{p-1} dx + \frac{1}{\tau}\int_\Omega u^p (f^{p-1})'(\alpha v - \lambda w) dx$$

$$= p\int_\Omega u^{p-1} u_t f^{p-1} dx + (p-1)\frac{\chi}{d_u \tau}\int_\Omega u^p f^{p-1}(\alpha v - \lambda w) dx$$

$$= p\int_\Omega u^{p-1}f^{p-1}(\nabla\cdot(d_u\nabla u+\chi u\nabla w))dx + p\mu_1\int_\Omega u^p f^{p-1}(1-u-a_1v)dx$$
$$+ (p-1)\frac{\chi}{d_u\tau}\int_\Omega u^p f^{p-1}(\alpha v-\lambda w)dx$$
$$= -p\int_\Omega d_u^{-1}\nabla(d_u u^{p-1}f^{p-1})(d_u\nabla u+\chi u\nabla w)dx$$
$$+ p\mu_1\int_\Omega u^p f^{p-1}(1-u-a_1v)dx + (p-1)\frac{\chi}{d_u\tau}\int_\Omega u^p f^{p-1}(\alpha v-\lambda w)dx.$$

Next we observe that

$$d_u\nabla(u^{p-1}f^{p-1}) = (p-1)u^{p-2}f^{p-1}(d_u\nabla u+\chi u\nabla w)$$

and we get

$$\frac{d}{dt}\int_\Omega u^p f^{p-1}dx = -p(p-1)d_u^{-1}\int_\Omega u^{p-2}f^{p-1}(d_u\nabla u+\chi u\nabla w)^2dx$$
$$+ p\mu_1\int_\Omega u^p f^{p-1}dx - p\mu_1\int_\Omega u^{p+1}f^{p-1}dx - p\mu_1 a_1\int_\Omega u^p f^{p-1}v dx$$
$$+ (p-1)\frac{\chi}{d_u\tau}\int_\Omega u^p f^{p-1}(\alpha v-\lambda w)dx\,.$$

Notice that thanks to (4.48) we get $f \le e^{\frac{\chi}{d_u}w_M}$ and

$$-u^{p+1}f^{p-1} \le -e^{-\frac{\chi}{d_u}w_M}u^{p+1}f^p$$

and finally we obtain the inequality

$$\frac{1}{p-1}\frac{d}{dt}\int_\Omega u^p f^{p-1}dx \le -\frac{p\mu_1}{(p-1)e^{\frac{\chi}{d_u}w_M}}\int_\Omega u^{p+1}f^p dx$$
$$+\left(\frac{p\mu_1}{p-1}+\frac{\chi}{d_u\tau}\alpha v_M\right)\int_\Omega u^p f^{p-1}dx. \quad \square$$

Next we proceed to prove a uniform $L^\infty$-bound for $u$.

**Lemma 4.6.** *There is a constant $u_M$ independent of time such that the first component of the solution to (4.34)–(4.36) satisfies*

$$\|u(t)\|_\infty \le \max\left\{f(w_M)\|u_0\|_\infty\,, \frac{\bar b+1}{\bar a}\right\} = u_M \quad \textit{for } t\in[0,T_{max}), \tag{4.50}$$

*where*

$$\bar a := \frac{\mu_1}{e^{\frac{\chi}{d_u}w_M}} \quad \textit{and} \quad \bar b := 2\mu_1 + \frac{\chi}{d_u\tau}\alpha v_M.$$

**Proof.** Let us denote

$$Y_p = \int_\Omega u^p f^{p-1} dx.$$

Then, for any $k > 0$ we have

$$\begin{aligned}\int_\Omega u^p f^{p-1}\, dx &= \int_{\{kauf\geq 1\}} u^p f^{p-1} dx + \int_{\{kauf<1\}} u^p f^{p-1}\, dx \\ &\leq ka \int_{\{kauf\geq 1\}} u^{p+1} f^p\, dx + (ka)^{1-p} \int_{\{kauf<1\}} u\, dx \\ &\leq ka \int_\Omega u^{p+1} f^p\, dx + (ka)^{1-p} \int_\Omega u\, dx.\end{aligned}$$

It follows that

$$-a \int_\Omega u^{p+1} f^p\, dx \leq \frac{-1}{k} \int_\Omega u^p f^{p-1}\, dx + k^{-p} a^{1-p} \int_\Omega u\, dx$$

and using (4.49) we have

$$\frac{1}{p-1} \frac{d}{dt} Y_p \leq \left( b - \frac{1}{k} \right) Y_p + k^{-p} a^{1-p} \int_\Omega u\, dx\,.$$

Choosing $k := (b+1)^{-1}$ and multiplying both sides of the inequality by $p-1$ we arrive at

$$\frac{d}{dt} Y_p \leq -(p-1) Y_p + k^{-p}(p-1) a^{1-p} \int_\Omega u\, dx\,.$$

Solving this inequality we get

$$Y_p \leq \max \left\{ Y_p(0)\,, (b+1)^p a^{1-p} \int_\Omega u dx \right\}$$

then, we find

$$Y_p^{\frac{1}{p}} \leq \max \left\{ Y_p(0)^{\frac{1}{p}}\,, (b+1) a^{-1+\frac{1}{p}} | \int_\Omega u dx|^{\frac{1}{p}} \right\}.$$

We now take limits as $p \to \infty$ to deduce that

$$\|u(t)\|_\infty \leq \max \left\{ f(w_M) \|u_0\|_\infty\,, (\overline{b}+1)(\overline{a})^{-1} \right\} = u_M$$

where

$$\overline{a} := \frac{\mu_1}{e^{\frac{\chi}{d_u} w_M}} \quad \text{and} \quad \overline{b} := 2\mu_1 + \frac{\chi}{d_u} \alpha v_M$$

and the proof ends. □

**Proof of Theorem 4.2.** Let us consider the system (4.38)–(4.40). Since $\|u(t)\|_\infty \le u_M$, $\|v(t)\|_\infty \le v_M$ and $\|w(t)\|_\infty \le w_M$ for $t \in [0\,, T_{max})$ we infer that there exist constants $z_M$ and $F_{2,M}$ such that $\|z(t)\|_\infty \le z_M$ and

$$\|F_2(z, v, w)\|_{L^\infty(0,T_{max}:L^\infty(\Omega))} \le F_{2,M}\,.$$

Using the classical semigroup estimates (see e.g. [6, Theorem 1.4.3])we obtain for $t \in [0\,, T_{max})$

$$\begin{aligned}\|v(t)\|_{X_p^\gamma} &\le \|v_0\|_{X_p^\gamma} + \int_0^t e^{-\delta(t-s)}(t-s)^{-\gamma} F_2(u(s), v(s), w(s))ds \\ &\le \|v_0\|_{X_p^\gamma} + \frac{k_\gamma}{1-\gamma} T_{max}^{1-\gamma} F_{2,M}\,.\end{aligned} \tag{4.51}$$

By (4.36) we have

$$\|w(t)\|_{X_p^\gamma} \le e^{-\lambda t}\|w_0\|_{X_p^\gamma} + \alpha \int_0^t e^{-\lambda(t-s)}\|v(t)\|_{X_p^\gamma} ds$$

and (4.43) along with $L^\infty$ bounds on $z$ and $v$ entail that there exists $F_{1,M} > 0$ such that

$$\|F_1(u, v, w)\|_{L^\infty(0,T_{max}:L^\infty(\Omega))} \le F_{1,M}$$

and similar argument as in (4.51) yields

$$\|z(t)\|_{X_p^\gamma} \le \|z_0\|_{X_p^\gamma} + \frac{k_\gamma}{1-\gamma} T_{max}^{1-\gamma} F_{1,M}\,.$$

Hence using Proposition 4.3 we deduce that $T_{max} = \infty$ and the solution is global in time. $\square$

**Remark 4.7.** Proposition 2.2 on the linear stablity of the steady state $P$ holds true also in the case $d_w = 0$. The only modification to the proof is related to the fact that in the parabolic–ODE case where the elliptic part of the system is degenerate. To prove the asymptotic stability we have to show that the spectrum of the linearizated operator is separated away from the imaginary axis. This result is easily deduced from Vieta formulae applied to the characteristic polynomial in the same way as in the proof of Theorem 3.1 in [26].

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