Mathematical Models and Methods in Applied Sciences
Vol. 26, No. 11 (2016) 2129–2162

DOI: 10.1142/S0218202516400108

Predator–prey model with diffusion and indirect prey-taxis

J. Ignacio Tello

Departamento de Matemática Aplicada,
ETSI Sistemas Informáticos,
Universidad Politécnica de Madrid,
28031 Madrid, Spain

Centro de Simulación Computacional,
Universidad Politécnica de Madrid,
Boadilla del Monte, 28660 Madrid, Spain
j.tello@upm.es

Dariusz Wrzosek*

Institute of Applied Mathematics and Mechanics,
University of Warsaw, Banacha 2,
02-097 Warszawa, Poland
D.Wrzosek@mimuw.edu.pl

Received 23 November 2015
Revised 29 May 2016
Accepted 6 July 2016
Published 22 September 2016
Communicated by M. Winkler

We analyze predator–prey models in which the movement of predator searching for prey is the superposition of random dispersal and taxis directed toward the gradient of concentration of some chemical released by prey (e.g. pheromone), Model II, or released from damaged or injured prey due to predation (e.g. blood), Model I. The logistic O.D.E. describing the dynamics of prey population is coupled to a fully parabolic chemotaxis system describing the dispersion of chemoattractant and predator's behavior. Global-in-time solutions are proved in any space dimension and stability of homogeneous steady states is shown by linearization for a range of parameters. For space dimension $N \leq 2$ the basin of attraction of such a steady state is characterized by means of nonlinear analysis under some structural assumptions. In contrast to Model II, Model I possesses spatially inhomogeneous steady states at least in the case $N = 1$.

Keywords: Predator–prey interactions; prey-taxis; chemotaxis; analytic semigroup; stability; bifurcation.

AMS Subject Classification: 92D40, 35K57, 35B40, 35B35

*Corresponding author

1. Introduction

Most of the models describing predator–prey interactions of spatially distributed populations have the structure of reaction–diffusion system in which reaction part is of Rosenzweig–MacArthur type,[27,25] (the generalization of the famous Lotka–Volterra system) and diffusive dispersion of each population is modeled by Laplace operator,[38,22] or cross-diffusion terms.[32] We refer the reader to an extensive survey by Jüngel,[18] which contains a large bibliography on the topic. However, there are many examples of predators whose foraging strategy of movement is the superposition of random dispersion with prey tactic pursuit. The mathematical model of predator–prey interaction with prey-taxis was proposed for the first time by Kareiva and Odell[19] and later extended by other authors.[12,4,28,1,21,34] Prey-taxis allows predators to search prey more actively and thus it is expected to eliminate more effectively local concentrations of prey. Contrary to a diffusion process that may give rise to Turing-like instabilities and pattern formation, prey-taxis in most cases tends to stabilize predator–prey interactions in such a way that predator activity in the long-time perspective transforms heterogeneous environment into homogeneous one (see Refs. 21 and 36). This is in some sense an opposite result to the case of Keller–Segel chemotaxis system where the chemotaxis may lead in space dimension $N \geq 2$ to the formation of aggregates or inhomogeneous space patterns (see Refs. 32 and 31). We also refer to the recent survey[3] on the role of chemotaxis in modeling pattern formation.

In this paper, we study two models describing "indirect" prey-taxis in the sense that the predator moves following the gradient of some chemical which indicates the presence of prey instead of moving directly toward the highest density of prey. In the Model IPT1, the role of chemoattractant detectable by an olfactory predator is played by a chemical substance released by prey already injured during capturing. Hunting of a carnivorous shark following the smell of blood is an example of such an indirect prey-taxis in a marine ecosystem (see e.g. Refs. 7, 13 and 40). We also refer to Lonnstedt *et al.*[23] reporting experimental studies on the role of different chemosensory cues in predator–prey interactions in marine ecosystems and Schuster[30] reporting tactic response of amoebas to chemicals produced by bacteria playing the role of prey. In the second Model IPT2, we consider the response of predator to some substances released by prey, as pheromones (kairomones) (see Refs. 39 and 17), chemical alarm cues,[9] sexual signals,[41] which may play the role of chemoattractant for the foraging predator. There is a vast biological literature devoted to chemosensory adaptations of predators for prey detection; though most of the attention has been paid to the role of chemical cues (kairomons) emitted by predators in development of defend adaptations in prey (see e.g. the survey by Ferrari[9]). We mention also that a model of chemotaxis with indirect signal production was studied recently,[35] in a different biological context.

To study the role of indirect prey-taxis in the formation of prey aggregations we propose two models which may be studied analytically and still indicate interesting features from the biological point of view. Our approach is based on the following main assumptions:

— We consider a generalist predator having in its diet a prey species which is of our interest.
— The life span of the predator is significantly longer than the one of the prey. It allows us to neglect the impact of the prey population on the growth rate of predator.
— The motility of prey is negligible with respect to predators motility.

Foraging of a planctivorous fish on zooplankton is an example of a situation in which such simplifications may be justified (see for instance Ref. 11). We propose a "minimal" model of dynamics of prey population accounting for the intraspecific competition which is modeled by the logistic term. The role of predator is reduced to foraging with search strategy being the superposition of random (diffusive) dispersal and chemotaxis. Hence, only changes in spatial distribution of predator population due to searching of food may cause changes in prey mortality in space due to local grouping of predators on aggregations of prey.

We denote by u and v the densities of predators and prey, respectively and assume that they occupy a bounded region $\Omega \subset \mathbb{R}^N$ with regular boundary $\partial\Omega$. The concentration of the chemoattractant released by prey is denoted by w.
We first consider

Model IPT1: Predator searching strategy is the superposition of random dispersion and directed movement toward the gradient of a chemical released by prey injured during capturing:

$$\begin{aligned} u_t &= d_u \Delta u - \operatorname{div}(\chi u \nabla w), && x \in \Omega, \ t > 0, \\ w_t &= d_w \Delta w - \mu w + \alpha v F_0(u), && x \in \Omega, \ t > 0, \\ v_t &= \lambda v \left(1 - \frac{v}{k}\right) - v F_0(u), && x \in \Omega, \ t > 0 \end{aligned}$$

with non-negative initial data and homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

where ν denotes the outer normal unitary vector at the boundary, d_u and d_w are the diffusion coefficients of the predators and chemoattractant, respectively, χ is the chemotactic sensitivity, μ is the decay rate of chemoattractant, λ is the growth rate in prey population and k its carrying capacity. We assume that F_0 is positive, bounded, smooth function and satisfies

$$F_0(0) = 0, \quad \lim_{u \to +\infty} F_0(u) = F_m.$$

The Monod function

$$F_0(u) = \frac{F_m u}{\kappa + u}, \quad \kappa, F_m > 0 \tag{1.1}$$

may serve as an example of such a function. The term $F_0(u)$ is interpreted as the mortality of prey due to the activity of predator with maximum mortality rate F_m.

On the other hand, in the classical theory of predator–prey interactions, the rate of prey consumption per predator is described as a function of prey and/or predator density named as the functional response. In our case, for the sake of simplicity the functional response is assumed to be proportional to v and decreasing with respect to u:

$$e(u,v) = \frac{F_m v}{\kappa + u}.$$

More advanced Holling's functional responses (see e.g. Ref. 25) taking into account the saturation of prey consumption for high prey densities may be used here. However it would make the analysis more complicated without substantial qualitative changes in the results.

The shape of the function e reflects the fact that due to an interference competition among spatially grouped predators the efficiency of a foraging individual predator decreases with the increase of the number of predators arriving at the same time at a patch of prey (see Ref. 8 where a similar simplistic formula was used to grasp the effect of predators grouping).

An alternative approach to taking into account the effect of local accumulation of foraging predators was proposed by Aiseba *et al.*[1] where a threshold density is introduced in such a way that the prey-taxis vanishes if the density of predators exceeds such a threshold.

The properties of IPT1 model are compared with that of the following Model IPT2 in which predators migrate toward the higher concentration of a chemoattractant secreted by the prey itself (the "smell" of prey) and the rest of the components of the Model IPT2 are the same as in Model IPT1.

Model IPT2:

$$\begin{aligned} u_t &= d_u \Delta u - \operatorname{div}(\chi u \nabla w), && x \in \Omega,\ \ t > 0, \\ w_t &= d_w \Delta w - \mu w + \alpha v, && x \in \Omega,\ \ t > 0, \\ v_t &= \lambda v \left(1 - \frac{v}{k}\right) - v F_0(u), && x \in \Omega,\ \ t > 0 \end{aligned}$$

with non-negative initial data and homogeneous Neumann boundary conditions. We introduce the following change of parameters and unknown variables to obtain a non-dimensional version of the IPT1 model. Assuming that the variables u, w and v have the same physical dimension and denoting by L the diameter of Ω (i.e. $L := \operatorname{diam}(\Omega)$) we set:

$$\tilde{u} = \frac{u}{\kappa}, \quad \tilde{w} := \frac{w}{k}, \quad \tilde{v} := \frac{v}{k},$$

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{d_u}{L^2} t, \quad \tilde{d}_w := \frac{d_w}{d_u}, \quad \tilde{\chi} = \frac{\chi k}{d_u}, \quad \tilde{F}_m := \frac{F_m L^2}{d_u},$$

$$\tilde{\lambda} = \frac{\lambda L^2}{d_u}, \quad \tilde{\mu} = \frac{\mu L^2}{d_u}, \quad \tilde{F}(\tilde{u}) = \frac{\tilde{F}_m \tilde{u}}{1 + \tilde{u}}.$$

Then the non-dimensional version of Model IPT1 reads:

$$\begin{aligned} \tilde{u}_t &= \Delta\tilde{u} - \operatorname{div}(\tilde{\chi}\tilde{u}\nabla\tilde{w}), && x \in \Omega,\ \ t > 0, \\ \tilde{w}_t &= \tilde{d}_w\Delta\tilde{w} - \tilde{\mu}\tilde{w} + \alpha\tilde{v}\tilde{F}_0(\tilde{u}), && x \in \Omega,\ \ t > 0, \\ \tilde{v}_t &= \tilde{\lambda}\tilde{v}(1 - \tilde{v}) - \tilde{v}\tilde{F}(\tilde{u}), && x \in \Omega,\ \ t > 0. \end{aligned}$$

To simplify the notation we drop the tilde in the aforementioned system and finally we obtain:

$$u_t = \Delta u - \operatorname{div}(\chi u \nabla w), \qquad x \in \Omega,\ \ t > 0, \tag{1.2}$$

$$w_t = d_w\Delta w - \mu w + \alpha v F(u), \qquad x \in \Omega,\ \ t > 0, \tag{1.3}$$

$$v_t = \lambda v(1 - v) - vF(u), \qquad x \in \Omega,\ \ t > 0 \tag{1.4}$$

with

$$F(u) = \frac{F_m u}{1 + u} \tag{1.5}$$

and boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega \tag{1.6}$$

and non-negative initial data

$$u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega. \tag{1.7}$$

Notice that the parameter α has a different interpretation in Model IPT2 than in Model IPT1, where it is dimensionless. In the latter case to get the non-dimensional version of IPT2 we set $\tilde{\alpha} = \frac{\alpha L^2}{d_u}$ and finally we obtain, after dropping the tilde:

$$u_t = \Delta u - \operatorname{div}(\chi u \nabla w), \quad x \in \Omega,\ \ t > 0, \tag{1.8}$$

$$w_t = d_w\Delta w - \mu w + \alpha v, \quad x \in \Omega,\ \ t > 0, \tag{1.9}$$

$$v_t = \lambda v(1 - v) - vF(u), \quad x \in \Omega,\ \ t > 0 \tag{1.10}$$

with

$$F(u) = \frac{F_m u}{1 + u}.$$

Systems of two parabolic equations with chemotactic terms coupled to an ordinary differential equations (O.D.E.) have been already studied in the literature (see for instance Ref. 3 and references therein), Refs. 5 and 16 where a parabolic–parabolic–O.D.E. system is presented to describe cancer invasion. Two chemotactic terms appears in the first equation (chemotaxis and hapotaxis) which presents a logistic growth, the system has been also studied in Ref. 21 where the convergence of the system to a parabolic–parabolic chemotaxis system is obtained. In Ref. 26, two biological species complete for the resources and follow a chemical gradient of a chemoattractant. The chemoattractant is a non-diffusive species produced by the biological species, the authors prove global existence and convergence to the homogeneous steady states under suitable assumptions in the coefficients.

The paper is organized as follows. Section 2 concerns the well-posedness of the models and existence of global-in-time solutions (Theorem 2.1). In Sec. 3, we point that there are constant steady states with positive components provided that the rate of growth of prey population prevails over the mortality caused by predator at the steady state. Then, using linearization about steady states we give conditions which warrant the local asymptotic stability of such constant steady states for both models (Theorem 3.1). It turns out that Model IPT1 has substantially different properties than Model IPT2 which does not present non-constant steady states while Model IPT1 have them. By using bifurcation theory we obtain that for sufficiently high values of chemotactic sensitivity or high values of initial mean value of predator density there are non-constant steady states at least in space dimension $N = 1$ (Proposition 3.3). In Sec. 4, we provide with effective conditions on the initial data and other model parameters for which solutions converge to the constant steady states (Theorem 4.1).

Along the paper we use the following notation

$$\langle f \rangle := \frac{1}{|\Omega|} \int_\Omega f$$

for the average value of function $f \in L^1(\Omega)$.

2. Global Existence of Solutions to IPT Models

In this section, we only consider Model IPT1, since arguments and results for Model IPT2 are analogous. We denote by $W^{k,p} := W^{k,p}(\Omega)$ the usual Sobolev space with the norm $\|\cdot\|_{k,p}$ and

$$W^{2,p}_\nu(\Omega) = \left\{ w \in W^{2,p}(\Omega) : \frac{\partial w}{\partial \nu}(x) = 0,\ x \in \partial\Omega \right\}.$$

Recall that for $p > N,\ W^{1,p} \subset C^\alpha(\Omega)$ for some $\alpha \in (0,1)$.

Theorem 2.1. *Assume that initial data are non-negative and for* $p > N, u_0, v_0 \in W^{1,p}(\Omega)$ *and* $w_0 \in W^{2,p}_\nu(\Omega)$*. Then there exists a unique solution* (u,w,v) *to* (1.2)–(1.4) *such that*

$$u, v \in C([0,\infty); W^{1,p}) \quad \textit{and} \quad w \in C\Big([0,\infty); W^{2,p}_\nu(\Omega)\Big)$$

and

$$u, w, v \geq 0 \quad \textit{on } \Omega \times [0,T). \tag{2.1}$$

Moreover, for any $T > 0, u, w \in C^{2,1}_{x,t}(\Omega \times (0,T))$ *and*

$$\sup_{t\in[0,\infty)} \max\{\|u(t)\|_\infty, \|w(t)\|_\infty,\ \|v(t)\|_\infty\} < \infty. \tag{2.2}$$

Proof. Let us define the space

$$X_T = \{\varphi \in C([0,T]; W^{1,p}) : \varphi(0) = 0, \|\varphi\|_{C([0,T];W^{1,p})} \leq R+1\},$$

where R is such that

$$\|u_0\|_{1,p} \leq R \tag{2.3}$$

and $T > 0$ will be specified later. We denote by $(e^{t\Delta})_{t\geq 0}$ the heat semigroup in the space $L^p(\Omega)$ with Neumann boundary condition. We first consider an auxiliary problem with F replaced by its extension $\tilde{F}$ defined by

$$\tilde{F}(u) = -F(-u) \quad \text{if } u < 0 \quad \text{and} \quad \tilde{F}(u) = F(u) \quad \text{otherwise.}$$

Notice that $\tilde{F}$ is a smooth globally Lipshitz function. We shall use the Banach fixed point theorem applied to the mapping $\Psi : X_T \mapsto X_T$ defined in the following way: for any $\tilde{u} \in X_T$ and fixed $v_0 \in W^{1,p}$ there is a unique $\hat{v} \in C([0,T]; W^{1,p})$ satisfying

$$v_t = \lambda v(1-v) - v\tilde{F}(\tilde{u}), \quad x \in \Omega,\ t \in [0,T]. \tag{2.4}$$

Indeed the right-hand side of (2.4) is a Lipschitz continuous function defined on the space $W^{1,p} \subset C^\alpha(\Omega)$ for $p > N$. Next, for $(\tilde{u}, \hat{v})$ there is a unique $\hat{w}$ such that

$$\hat{w}(t) = e^{(d_w\Delta-\mu)t}w_0 + \int_0^t e^{(d_w\Delta-\mu)(t-s)}\alpha\hat{v}(s)\tilde{F}(\tilde{u}(s))ds \tag{2.5}$$

and finally applying the parabolic regularity to $\hat{w}$ we define

$$\Psi(\tilde{u}) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}(\chi\nabla\tilde{u}(s)\nabla\hat{w}(s) + \tilde{u}(s)\Delta\hat{w}(s))ds. \tag{2.6}$$

First we shall show that for T small enough:

$$\|\Psi(\tilde{u})\|_{C([0,T];W^{1,p})} \leq R + 1 \tag{2.7}$$

for any $\tilde{u} \in X_T$. Indeed using the fact that F has a bounded derivative we infer that there is $M > 0$ such that

$$\sup_{t\in[0,T]} \|\hat{v}\|_{1,p} < M \tag{2.8}$$

uniformly with respect to $\tilde{u} \in X_T$. Then (1.5) and (2.8) imply that there is a constant $K > 0$ which depends on M, F and R such that

$$\|\alpha\hat{v}\tilde{F}(\tilde{u})\|_{C([0,T];W^{1,p})} \leq K \tag{2.9}$$

uniformly for $\tilde{u} \in X_T$. We shall use the following semigroup estimate[24]: there is a constant C such that for $k \geq l, l, k \in \{0,1,2\}$:

$$\begin{gathered}\|e^{(d_w\Delta-\mu)t}\varphi\|_{k,q} \leq Ct^{-\frac{k-l}{2}}\|\varphi\|_{l,q}, \\ \varphi \in W^{l,q} \quad \text{if } k = 1 \quad \text{and} \quad \varphi \in W^{l,q}_\nu \quad \text{for } k = 2.\end{gathered} \tag{2.10}$$

It then follows from (2.10) that for $t > 0$:

$$\|\hat{w}(t)\|_{2,p} \leq C\|w_0\|_{2,p} + \int_0^t (t-s)^{-1/2}C\|\alpha\hat{v}(s)\tilde{F}(\tilde{u}(s))\|_{1,p}ds. \tag{2.11}$$

From (2.9), and singular Gronwall's inequality[14] we deduce that there is a constant $K_1 = K_1(K, C, T)$ such that

$$\sup_{t\in[0,T]} \|\hat{w}\|_{2,p} \leq K_1. \tag{2.12}$$

Now we use again (2.10) in (2.6) to obtain

$$\begin{aligned} \|\Psi(\tilde{u})\|_{1,p} &\leq C\|u_0\|_{1,p} + \int_0^t (t-s)^{1/2} C\|(\chi\nabla\tilde{u}(s)\nabla\hat{w}(s) + \tilde{u}(s)\Delta\hat{w}(s))\|_{0,p} ds \\ &\leq R + 4T^{1/2}CRK_1, \end{aligned}$$

where we have used (2.12) and (2.3). Now we may fix T small enough so that $4T^{1/2}CRK_1 \leq 1$ thus Ψ is well defined. Using similar arguments one can prove that for suitably chosen T, mapping Ψ is also contractive. Thus there exists a fixed point $u \in X_T$ of Ψ which uniquely determines v and w by (2.4) and (2.5), respectively. From the regularity theory of parabolic equations it follows that

$$u, w \in C^{2,1}_{x,t}(\bar{\Omega} \times (0,T)).$$

The uniqueness of solutions results easily from the regularity of w and Lipschitz continuity of both function $\tilde{F}$ and the right-hand side of (1.4). A standard extension argument implies the existence of the maximal existence time $T_{\max}$. Let us consider the equation for u, which reads

$$u_t - \Delta u + \chi\nabla\cdot(u\nabla w) = 0. \tag{2.13}$$

Then we split the nonlinear second-order terms to obtain

$$u_t - \Delta u + \chi\nabla u\nabla w = -u\chi\Delta w. \tag{2.14}$$

Since $w \in C^{2,1}_{x,t}(\Omega \times (0, T_{\max}))$ we have that

$$\tilde{u}_t - \Delta\tilde{u} + a(x,t)\nabla\tilde{u} = b(x,t)\tilde{u}, \tag{2.15}$$

where $a(x,t) \in C^{1,0}_{x,t}(\Omega \times (0, T_{\max}))$ and $b \in C^0(\Omega \times (0, T_{\max}))$. It follows from parabolic maximum principle and non-negativity of the initial data that

$$u \geq 0. \tag{2.16}$$

The non-negativity of v follows from the representation

$$v(x,t) = v_0 \exp\left\{\int_0^t h(x,s)ds\right\},$$

where $h = h(t,x)$ is a bounded function. The non-negativity of w results again from the maximum principle. The non-negativity of the solution results in $\tilde{F}(u) = F(u)$ and owing to the uniqueness of solution we infer the existence of the solution to the original problem.

We are in a position to prove that the solution is global-in-time. To this end we first deduce, using maximum principle, that $\|w(t)\|_\infty$ is uniformly bounded for all

$t \in [0, T_{\max})$ because it holds $vF(u) \in L^\infty(\Omega \times (0, T_{\max}))$. By the classical parabolic L^q-regularity theory we have that for any $q \geq 2$:

$$w \in L^q(0, T_{\max} : W^{2,q}(\Omega)) \cap W^{1,q}(0, T_{\max} : L^q(\Omega)). \tag{2.17}$$

It follows by Lemma 3.3, p. 80 in Ref. 20, that for $q > N + 2$ we have $\nabla w \in [L^\infty(\Omega \times (0, T_{\max}))]^N$. Using Moser–Alikakos iterative method[2] applied to the scalar parabolic equation (2.13) one can prove that there is a constant C_1 independent on time such that

$$\sup_{t\in[0,T_{\max})} \|u(t)\|_\infty \leq C_1 \max\left\{ \sup_{t\in[0,T_{\max})} \|u(t)\|_1, 1 \right\}. \tag{2.18}$$

Many variants of the Moser–Alikakos method are available in the literature. Here we give only a short sketch adjusted to our case. Upon multiplying (2.13) by u^{2^k-1}, $k \geq 1$, and integrating over Ω one obtains

$$\begin{aligned} \frac{d}{dt}\frac{1}{2^k}\int_\Omega u^{2^k} &= -(2^k-1)2^{2-2k}\int_\Omega \left|\nabla u^{2^{k-1}}\right|^2 \\ &\quad + \chi(2^k-1)2^{1-k}\|\nabla w\|_\infty \int_\Omega u^{2^{k-1}}\left|\nabla u^{2^{k-1}}\right| \\ &\leq -\int_\Omega \left|\nabla u^{2^{k-1}}\right|^2 + 2\chi\|\nabla w\|_\infty \left(\int_\Omega \left|u^{2^{k-1}}\right|^2\right)^{1/2} \left(\int_\Omega \left|\nabla u^{2^{k-1}}\right|^2\right)^{1/2} \end{aligned}$$

and next using the Gagliardo–Nirenberg inequality one follows lines of proof of Lemma 9.3.1 in Ref. 6.

Notice that by the Neumann boundary condition it holds $\int_\Omega u(t,x)dx = \int_\Omega u_0$ for $t \in [0, T_{\max})$, hence by (2.18) $u \in L^\infty(\Omega \times (0, T_{\max}))$. Thanks to (2.17) we have enough regularity of the coefficients in the parabolic equation (2.14) to apply Theorem 1.11 in Ref. 20, and infer that also $\nabla u \in L^\infty(\Omega \times (0, T_{\max}))$. Directly from (2.4) we obtain that

$$v \in C([0, T_{\max}) : W^{1,p}(\Omega))$$

and hence in light of (2.11):

$$\sup_{t\in[0,T_{\max})} \|w(t)\|_{2,p} < \infty.$$

Finally we deduce that $T_{\max} = \infty$ and the solution is global-in-time. Next we deduce from (2.18) that

$$\sup_{t\in[0,\infty)} \|u(t)\|_\infty < \infty. \tag{2.19}$$

By comparison with logistic O.D.E. applied to (1.4) we obtain that v is uniformly bounded in L^∞ and

$$0 \leq v(\cdot, t) \leq \max\left\{1, \left(1 + \left(\|v_0\|_\infty^{-1} - 1\right)e^{-\mu_2 t}\right)^{-1}\right\} \quad \text{for } t \in [0, \infty). \tag{2.20}$$

On integrating (1.3) over Ω and using (2.20) we easily obtain that

$$\sup_{t\in[0,\infty)} \int_\Omega w(t) < \infty$$

and now we may use the same argument as in (2.18) to obtain

$$\sup_{t\in[0,\infty)} \|w(t)\|_\infty \leq \infty. \tag{2.21}$$

To complete the proof notice that (2.19)–(2.21) imply (2.2). □

3. Steady States and Linearization

In this section, we study the steady states to Model IPT1 (1.2)–(1.4) and to Model IPT2 (1.8)–(1.10) assuming that $\int_\Omega u_0 := M > 0$. It follows from the non-flux boundary condition (1.6) that

$$\langle u(t)\rangle = \langle u_0\rangle = \frac{M}{|\Omega|}, \quad \text{for } t > 0. \tag{3.1}$$

If $F(\bar{u}) < \lambda$ then it is easy to see that there is only one constant steady state $P_1^1 = (\bar{u}, \bar{w}, \bar{v})$ to system IPT1 (1.2)–(1.4) with positive components:

$$\bar{u} := \langle u\rangle, \quad \bar{w} = \frac{\alpha}{\mu}\left(1 - \frac{F(\langle u\rangle)}{\lambda}\right) F(\langle u\rangle), \quad \bar{v} = 1 - \frac{F(\langle u\rangle)}{\lambda} \tag{3.2}$$

and there is only one homogeneous steady state $P_1^2 = (\bar{u}, \bar{w}, \bar{v})$ to system IPT2 (1.8)–(1.10) with

$$\bar{u} = \langle u\rangle, \quad \bar{w} = \frac{\alpha}{\mu}\left(1 - \frac{F(\langle u\rangle)}{\lambda}\right), \quad \bar{v} = 1 - \frac{F(\langle u\rangle)}{\lambda}. \tag{3.3}$$

There is also a trivial steady state P_0 for both models:

$$\bar{u} = \langle u\rangle, \quad \bar{w} = \bar{v} = 0, \tag{3.4}$$

which is a unique space of homogeneous steady state provided $F(\langle u\rangle) \geq \lambda$.

Linearization at a homogeneous steady state $(\bar{u}, \bar{w}, \bar{v})$ to Model IPT1 leads to the following eigenvalue problem:

$$\Delta\varphi - \chi\bar{u}\Delta\psi = \sigma\varphi, \tag{3.5}$$

$$d_w\Delta\psi - \mu\psi + \alpha\bar{v}F'(\bar{u})\varphi + \alpha F(\bar{u})\eta = \sigma\psi, \tag{3.6}$$

$$-\bar{v}F'(\bar{u})\varphi + (F(\bar{u}) - \lambda)\eta = \sigma\eta, \tag{3.7}$$

where $(\varphi, \psi, \eta) \in X_0 \times X \times Y$ and:

$$X_0 = \left\{\varphi \in W^{2,p}(\Omega) : \frac{\partial\varphi}{\partial\nu} = 0, \int_\Omega \varphi(x)dx = 0\right\},$$

$$X = \left\{\psi \in W^{2,p}(\Omega) : \frac{\partial\psi}{\partial\nu} = 0\right\}, \quad Y = L^2(\Omega).$$

Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence of eigenvalues of operator $-\Delta$ with homogeneous Neumann boundary conditions defined on X:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \tag{3.8}$$

and $\{w_n\}_{n=0}^{\infty}$ the corresponding system of orthonormal eigenfunctions in $L^2(\Omega)$. Let us define the matrix

$$A_n = \begin{bmatrix} -\lambda_n & \chi\bar{u}\lambda_n & 0 \\ \alpha\bar{v}f_1 & -(d_w\lambda_n + \mu) & \alpha f \\ -\bar{v}f_2 & 0 & r \end{bmatrix}, \tag{3.9}$$

where $f = F(\bar{u}), f_1 = f_2 = F'(\bar{u})$ in Model IPT1 and $f = 1, f_1 = 0, f_2 = F'(\bar{u})$ in the case of Model IPT2 and $r = \lambda - F(\bar{u}) - 2\bar{v}\lambda$ in both cases.

Proposition 3.1. *A complex number σ is an eigenvalue of* (3.5)–(3.7) *if and only if there exists $n \geq 1$ such that σ is an eigenvalue of matrix* (3.9) *or for $n = 0$, $\sigma \in \{-\mu, r\}$. Moreover, spectrum of linear operator in* (3.5)–(3.7) *consist only of eigenvalues.*

Proof. Let $\varphi = \sum_{n=0}^{\infty} \varphi_n w_n, \psi = \sum_{n=0}^{\infty} \psi_n w_n, \eta = \sum_{n=0}^{\infty} \eta_n w_n$. On multiplying each equation in (3.5)–(3.7) by w_n and integrating over Ω using the boundary condition we obtain for $n \geq 1$ the following eigenvalue problem

$$A_n \begin{bmatrix} \varphi_n \\ \psi_n \\ \eta_n \end{bmatrix} = \sigma \begin{bmatrix} \varphi_n \\ \psi_n \\ \eta_n \end{bmatrix},$$

where A_n was defined in (3.9). For $n = 0$ as a consequence of the condition $\int_\Omega \varphi = 0$ we have $\varphi_0 = 0$ which leads to the eigenvalue problem

$$\begin{bmatrix} -\mu & -\alpha f \\ 0 & r \end{bmatrix} \begin{bmatrix} \psi_0 \\ \eta_0 \end{bmatrix} = \sigma \begin{bmatrix} \psi_0 \\ \eta_0 \end{bmatrix}.$$

Using the resolvent equation and testing each equation by w_n we easily deduce that any σ which is not an eigenvalue belongs to the resolvent set which completes the proof. □

The characteristic polynomial of A_n reads:

$$p_n(\sigma) = \sigma^3 + a_2(\lambda_n)\sigma^2 + a_1(\lambda_n)\sigma + a_0(\lambda_n), \tag{3.10}$$

where

$$a_2(\lambda_n) = \lambda_n + (\mu + d_w\lambda_n) - r,$$
$$a_1(\lambda_n) = \det \tilde{A} - \lambda_n r - r(d_w\lambda_n + \mu),$$
$$a_0(\lambda_n) = -r \det \tilde{A} + \bar{v}\beta_0\lambda_n,$$

where

$$\det \tilde{A} = \lambda_n(\mu + d_w\lambda_n) - \bar{v}Y_0\lambda_n \tag{3.11}$$

is the second main minor of matrix A_n with:

$$Y_0 := \alpha\chi\bar{u}f_1, \tag{3.12}$$

$$\beta_0 := \alpha\chi\bar{u}f_2 f. \tag{3.13}$$

We are in a position to formulate the main theorem of this section on the linear stability of the steady states.

Theorem 3.1. *Stability conditions of steady states* P_1^1, P_1^2 *and* P_0 *are the following*:

(1) *Steady state* P_1^1 *in Model* IPT1 *is locally asymptotically stable if*

$$\frac{\chi\alpha\bar{u}F'(\bar{u})}{\lambda} < (1+d_w)\min\left(\frac{2\mu}{\mu+F(\bar{u})}, 1\right). \tag{3.14}$$

(2) *Steady state* P_1^2 *in Model* IPT2 *is locally asymptotically stable if*

$$\frac{\chi\alpha\bar{u}F'(\bar{u})}{\lambda} < 2(1+d_w)\mu. \tag{3.15}$$

(3) *Steady state* P_0 *is locally asymptotically stable if* $\lambda < F(\bar{u})$.

In each of the above cases there exists $\delta_0 > 0$ *such that if* σ *is an eigenvalue to* (3.5)–(3.7) *then* $\operatorname{Re}\sigma < -\delta_0 < 0$.

Steady state P_0 *is unstable provided* $\lambda \geq F(\bar{u})$. *There is* $K > 0$ *such that steady states* P_1^1 *and* P_1^2 *are unstable provided*

$$\frac{\chi\alpha\bar{u}}{\lambda} > K. \tag{3.16}$$

Proof. We first consider points P_1^1 and P_1^2. It follows from Routh–Hurwitz theorem (see e.g. Appendix B.1 in Ref. 25) that roots of (3.10) have negative real parts if and only if

$$a_2(\lambda_n) > 0, \quad a_0(\lambda_n) > 0, \quad a_1(\lambda_n)a_2(\lambda_n) > a_0(\lambda_n). \tag{3.17}$$

Notice that $a_2(\lambda_n) > 0$ is satisfied as long as $r < 0$. For the case of P_1^1 and P_1^2 we have $r = F(\bar{u}) - \lambda < 0$ and $\bar{v} = \frac{-r}{\lambda}$. By straightforward calculation we check that $a_0(\lambda_n) = \lambda\alpha\chi\bar{u}\bar{v}F'(\bar{u})\lambda_n > 0$. Next we define

$$\begin{aligned} H(\lambda_n) &= a_1(\lambda_n)a_2(\lambda_n) - a_0(\lambda_n) \\ &= b_3\lambda_n^3 + b_2\lambda_n^2 + b_1\lambda_n + b_0, \end{aligned}$$

where

$$\begin{aligned} b_3 &= d_w + d_w^2, \\ b_2 &= \mu(1+2d_w) + r(1+d_w)(-(1+d_w) + Y_0/\lambda), \\ b_1 &= \mu^2 + r^2(1+d_w) - r(2\mu(1+d_w) - Y_0\mu/\lambda - \beta_0/\lambda), \\ b_0 &= -r\mu^2 + r^2\mu. \end{aligned} \tag{3.18}$$

Notice that $b_3 > 0, b_0 > 0$ and it is easy to check that $b_2 > 0$ for any $r < 0$ provided

$$Y_0/\lambda < 1 + d_w. \tag{3.19}$$

Moreover, $b_1 > 0$ for any $r < 0$ provided

$$Y_0\mu/\lambda + \beta_0/\lambda < 2\mu(1 + d_w)$$

and using definition of Y_0 and β_0 it follows that

$$\frac{\chi\alpha\bar{u}(\mu f_1 + f_2 f)}{\lambda} < 2\mu(1 + d_w) \tag{3.20}$$

and consequently (3.14) follows from (3.20), (3.19) and definition of Y_0. Observe that in the case of the steady state P_1^2 in Model IPT2 we have $r = F(\bar{u}) - \lambda < 0$ and $Y_0 = 0$ as the consequence of condition $f_1 = 0$. It is easy to check that in this case $b_3 > 0, b_2 > 0, b_0 > 0$ and using (3.20) $b_1 > 0$ provided

$$\frac{\chi\alpha\bar{u}f_2 f}{\lambda} < 2\mu(1 + d_w),$$

whence (3.15) follows.

Finally in the case of P_0 we have that

$$A_n = \begin{bmatrix} -\lambda_n & \chi\bar{u}\lambda_n & 0 \\ 0 & -(d_w\lambda_n + \mu) & \alpha f \\ 0 & 0 & r \end{bmatrix}$$

and the eigenvalues of A_n are $-\lambda_n, -(d_w\lambda_n + \mu)$ and r. Since $\lambda_n > 0$ for any $n \geq 0$ we have that $-\lambda_n$ and $-(d_w\lambda_n + \mu)$ are negative.

- If $F(\bar{u}) > \lambda$ then $r = \lambda - F(\bar{u}) < 0$ and the three eigenvalues of A_n are negative, which proves (3).
- If $F(\bar{u}) = \lambda$ then $r = 0$, which precludes the stability of P_0.
- If $F(\bar{u}) < \lambda$ then $r > 0$ and then also precludes stability.

Finally we shall show that the set of eigenvalues is separated away from the imaginary axis. To this end assume that $\sigma_1(n), \sigma_2(n), \sigma_3(n)$ are roots of polynomial $p_n(\sigma)$ in (3.10). We have already checked that for all $n \geq 1$ and $i = 1, 2, 3$ $\operatorname{Re}\sigma_i(n) < 0$. By Viete's formula we have:

$$\sigma_1(n) + \sigma_2(n) + \sigma_3(n) = -a_2(\lambda_n) = -(1 + d_w)\lambda_n + o(\lambda_n), \tag{3.21}$$

$$\sigma_1(n) \cdot \sigma_2(n) \cdot \sigma_3(n) = -a_0(\lambda_n) = -rd_w\lambda_n^2 + o(\lambda_n^2), \tag{3.22}$$

where we adopt the following notation $\lim_{\gamma_n\to\infty} \frac{o(\gamma_n)}{\gamma_n} = 0$ and $\lim_{\gamma_n\to\infty} \frac{O(\gamma_n)}{\gamma_n} =$ const. It follows from (3.21)–(3.22) that exactly two roots, say $\sigma_2(n)$ and $\sigma_3(n)$ satisfy $\sigma_2(n) = O(\lambda_n)$ and $\sigma_3(n) = O(\lambda_n)$ and next we deduce from (3.22) that there is $\bar{\sigma}$ such that

$$\lim_{n\to\infty} \sigma_1(n) = \bar{\sigma} < 0.$$

Hence there exists $\delta_0 > 0$ such that for any $n \geq 0$ and $i = 1, 2, 3$, $\operatorname{Re}\sigma_i(n) < -\delta_0 < 0$. Now we are in a position to use Chap. 11 in Ref. 33, to deduce the local linear stability of steady states P_1^1, P_1^2 and P_0 under conditions listed in the statement of the theorem.

To prove instability of P_1^1 it remains to show using again the Routh–Hurwitz criterion (3.17) that for fixed λ_n, and suitable values of parameters the coefficient $a_1(\lambda_n)$ may be negative. To this end notice that the negative component in (3.11) reads

$$-\bar{v}Y_0\lambda_n = \left(\frac{\alpha\chi\bar{u}}{\lambda}\right) rF'(\bar{u})\lambda_n$$

and for sufficiently big $\frac{\alpha\chi\bar{u}}{\lambda}$ there is $a_1(\lambda_n) < 0$. To prove instability of P_1^2 it remains to notice that in this case $Y_0 = 0$ and using (3.18) we have $b_1 < 0$ provided

$$\frac{\beta_0}{\lambda} = \left(\frac{\alpha\chi\bar{u}}{\lambda}\right) F(\bar{u})F'(\bar{u})$$

is big enough. □

To show that there are non-constant steady states to Model IPT1 we shall consider an auxiliary system of two nonlinear elliptic equations. Notice that using (1.4) the steady state problem for system (1.2)–(1.4) may be reduced to the system of two equations:

$$\Delta\bar{u} - \operatorname{div}(\bar{u}\chi\nabla\bar{w}) = 0, \quad x \in \Omega, \tag{3.23}$$

$$d_w\Delta\bar{w} - \mu\bar{w} + \alpha\left(1 - \frac{F(\bar{u})}{\lambda}\right)F(\bar{u}) = 0, \quad x \in \Omega \tag{3.24}$$

with no-flux boundary condition. Let us denote

$$\Gamma(u) = \left(1 - \frac{F(u)}{\lambda}\right)F(u)$$

and

$$\gamma := \Gamma'(\bar{u}) = \left(1 - \frac{2F(\bar{u})}{\lambda}\right)F'(\bar{u}). \tag{3.25}$$

If $F(\bar{u}) \leq \lambda$ then any steady state of (3.23)–(3.24) defines the first two component of steady state of Model IPT1 and vice versa. Linearization of system (3.23)–(3.24) at the constant steady state $(\bar{u}, \bar{w})$ leads to the following eigenvalue problems:

$$\Delta\phi - \chi\bar{u}\Delta\psi = \sigma\phi, \tag{3.26}$$

$$d_w\Delta\psi - \mu\psi + \alpha\gamma\phi = \sigma\psi, \tag{3.27}$$

where $(\phi, \psi) \in X_0 \times X$ and

$$X_0 = \left\{\phi \in W^{2,p}(\Omega) : \frac{\partial\phi}{\partial\nu} = 0, \int_\Omega \phi(x)dx = 0\right\},$$

$$X = \left\{\phi \in W^{2,p}(\Omega) : \frac{\partial\psi}{\partial\nu} = 0\right\}.$$

Let $\{\lambda_n\}_{n=1}^\infty$ be the sequence of eigenvalues of $-\Delta$ defined in (3.8). Let us define the matrix

$$B_n = \begin{bmatrix} -\lambda_n & \chi\bar{u}\lambda_n \\ \alpha\gamma & -\lambda_n d_w - \mu \end{bmatrix}. \tag{3.28}$$

Proposition 3.2. *A complex number* σ *is an eigenvalue to problem* (3.26)–(3.27) *if and only if there exists* $n \geq 1$ *such that* σ *is the eigenvalue of matrix* B_n *or* $\sigma = -\mu$. *Moreover*, $\operatorname{Re} \sigma < 0$ *if and only if*

$$\lambda_1 > \frac{\alpha\gamma\chi\bar{u} - \mu}{d_w}. \tag{3.29}$$

Proof. The first part of the proposition is based on similar arguments as in the proof of Theorem 3.1 (see also Ref. 29 or Ref. 37). It is easy to verify that for $n \geq 1$ it holds

$$\operatorname{tr} B_n = -(1 + d_w)\lambda_n - \mu < 0 \quad \text{and} \quad \det B_n = \lambda_n(\lambda_n d_w + \mu - \gamma\alpha\chi\bar{u})$$

and $\det B_n > 0$ for all $n \geq 1$ if and only if (3.29) holds. □

Notice that the stationary problem (3.23)–(3.24) corresponds to the following evolutionary system:

$$U_t = \Delta U - \operatorname{div}(U\chi\nabla W), \qquad x \in \Omega,\ t > 0, \tag{3.30}$$

$$W_t = d_w\Delta W - \mu W + \alpha\left(1 - \frac{F(U)}{\lambda}\right)F(U), \quad x \in \Omega,\ t > 0, \tag{3.31}$$

which has one constant steady state $(\bar{u}, \bar{w})$ with positive components. This system may be interpreted as a simplified version of the original model in which prey population stabilizes fast enough so that it is justified to assume $v_t \approx 0$. Moreover, letting $\lambda \to \infty$ we obtain the chemotaxis model with nonlinear signal kinetics which was mentioned in Ref. 15. It corresponds to the case where the prey density is close to the steady state $v \equiv 1$ and the total system is reduced to the first two equations. Standard linearized stability analysis for dynamical system generated by (3.30) and (3.31) leads to the following conclusion.

Corollary 3.1. *If* (3.29) *holds then constant steady state* $(\bar{u}, \bar{w})$ *is locally asymptotically stable for the dynamical system generated by* (3.30) *and* (3.31).

Remark 3.1. It is worth noticing that (3.29) is satisfied whenever $\gamma < 0$ and owing to (3.25) it is equivalent to the condition $F(\bar{u}) > \frac{\lambda}{2}$. From the point of view of the linearized stability our case $\gamma < 0$, $\chi > 0$ is analogous to the case of chemorepulsion i.e. $\chi < 0$ and $\gamma > 0$.

If $\gamma > 0$ then it is convenient to choose χ as a bifurcation parameter and then condition (3.29) for stability of $S_0 = (\bar{u}, \bar{w})$ reads

$$\chi < \chi_1 := \frac{\lambda_1 d_w + \mu}{\gamma\alpha\bar{u}}.$$

We are now in a position to adapt the result from Theorem 2.4 of Ref. 37, derived in the case of one space dimension.

Proposition 3.3. *Assume* $N = 1$ *and* $M > 0$. *Then for any* $\chi > \chi_1$ *there are non-constant steady states with the mass* M.

It is proved in Theorem 2.4 of Ref. 37, using bifurcation theory that each component of such a non-constant steady state may be a monotone increasing or decreasing function. However using homogeneous Neumann boundary condition and periodic extension or reflection of a monotone function a non-monotone steady state may be constructed for appropriately enlarged domain Ω.

It turns out that Model IPT1 has substantially different properties from Model IPT2 as the latter does not have non-constant steady states. Indeed notice first that in the case of Model IPT2 reduction to the system of two equations leads to:

$$\Delta\bar{u} - \operatorname{div}(\bar{u}\chi\nabla\bar{w}) = 0, \quad x \in \Omega, \tag{3.32}$$

$$d_w\Delta\bar{w} - \mu\bar{w} + \alpha\left(1 - \frac{F(\bar{u})}{\lambda}\right) = 0, \quad x \in \Omega \tag{3.33}$$

and after the linearization about constant steady state with positive components we obtain the same matrix as in (3.28) with only difference in $\gamma = -\frac{\alpha}{\mu}F'(\bar{u}) < 0$ instead of (3.25). In light of (3.29) it follows that the space-homogeneous steady state is linearly locally asymptotically stable for all set of parameters and in particular it does not lose stability when χ is big enough. On the other hand it turns out that the solution to (3.32) and (3.33) is uniquely determined. Indeed using standard argument we infer from (3.32) that there is $\varrho > 0$ such that

$$u = \varrho e^{\chi w}.$$

Then any nonzero steady state of (3.32) and (3.33) satisfies the following semilinear elliptic equation

$$-\Delta w + \mu w + R(w) = 0 \quad \text{on } \Omega, \tag{3.34}$$

with no-flux boundary condition where $R(w) = \alpha\left(\frac{F(\varrho e^{\chi w})}{\lambda} - 1\right)$. Uniqueness of solutions to (3.34) results from the fact that $w \mapsto \mu w + R(w)$ for $w > 0$ is a strictly increasing function and classical arguments for monotone operators may be applied.

4. Asymptotic Behavior

In this section, we study the asymptotic behavior of solutions to Model IPT1, (1.2)–(1.4) and Model IPT2, (1.8)–(1.10) for bounded and open domain with regular boundary $\Omega \subset \mathbb{R}^N$ for $N \leq 2$. We assume in both cases that

$$\lambda > F_m. \tag{4.1}$$

The previous assumption guarantees the existence of a unique positive steady state denoted by $P_1^1 = (\bar{u}, \bar{w}, \bar{v})$ for Model IPT1 (P_1^2 for Model IPT2). The steady state is an attractor for non-negative initial data provided the initial conditions and parameters satisfy additional conditions listed below:

- the initial data u_0 is bounded and non-negative, i.e. satisfies

$$0 \leq u_0(x) \leq \|u_0\|_{L^\infty(\Omega)} < \infty, \tag{4.2}$$

- there exists a positive constant $\underline{v}_0 > 0$ such that the initial data v_0 satisfies

$$\underline{v}_0 \leq v_0(x) \leq 1, \tag{4.3}$$

- bounded and positive initial data w_0 satisfies

$$0 \leq w_0(x) \leq w_{\max} < \infty, \tag{4.4}$$

- furthermore, for the Model IPT1, u_0 satisfies

$$\max\left\{\frac{c^*\langle u_0\rangle}{2d_w}, \frac{\langle u_0\rangle + 2}{\mu}\right\} \frac{8\alpha^2 F_m^2 \chi^2}{\mu}\left(1 + \frac{F_m^2}{\lambda \min\{\lambda \underline{v}_0, (\lambda - F_m)\}}\right) < 1, \tag{4.5}$$

- and for the Model IPT2, u_0 satisfies

$$\max\left\{\frac{\chi^2 c^*\langle u_0\rangle}{2d_w}, \frac{\chi^2(\langle u_0\rangle + 2)}{\mu}\right\} \frac{8\alpha^2}{\mu\lambda} \frac{F_m^2}{\min\{\lambda \underline{v}_0, (\lambda - F_m)\}} < 1, \tag{4.6}$$

where c^* is some constant which depends on constants in the Poincaré inequality and the Gagliardo–Nirenberg inequality.

Theorem 4.1. *Under assumptions* (4.1)–(4.4) *and space dimension* $N \leq 2$, u_0 *satisfying* (4.5) *for the* IPT1 *model or* (4.6) *for the* IPT2 *model, the solution satisfies*:

$$u \to \langle u_0\rangle \quad \text{in } L^p(\Omega) \text{ as } t \to \infty,$$
$$v \to \bar{v} \quad \text{in } L^p(\Omega) \text{ as } t \to \infty,$$
$$w \to \bar{w} \quad \text{in } L^p(\Omega) \text{ as } t \to \infty$$

for any $p \in [1, \infty)$, *where* $\bar{w}$ *and* $\bar{v}$ *are defined in* (3.2) *and* (3.3) *for Model* IPT1 *and Model* IPT2, *respectively.*

The proof of the theorem is divided into several steps. We first obtain a set of *a priori* estimates and using an energy method we obtain the wished stability result. The computations to obtain the estimates for u and v are the same for both models, therefore we only present them once.

Lemma 4.1. *Under assumptions of Theorem* 4.1, (4.1) *and* (4.3) *we have that*

$$1 \geq v \geq \min\left\{\underline{v}_0, 1 - \frac{F_m}{\lambda}\right\}.$$

Proof. Since $F(u) = F_m u/(1+u) \leq F_m$ we have that

$$v_t \geq v(\lambda - F_m - \lambda v),$$

maximum principle and assumption (4.3) give us the second inequality. To end the proof, we notice that $F(u) \geq 0$ and we solve the inequality

$$v_t \leq \lambda v(1 - v)$$

to obtain

$$v \leq \max\{1, \sup\{v_0\}\} = 1. \qquad \square$$

Lemma 4.2. *Under assumptions of Theorem* 4.1 *and* (4.4):

- *For the Model* IPT1, *we have that*

$$0 \le w \le \max\{\|w_0\|_{L^\infty(\Omega)}, \alpha F_m/\mu\}$$

for any $t > 0$.
- *For the Model* IPT2, *we have that*

$$0 \le w \le \frac{1}{\mu}\max\{\|w_0\|_{L^\infty(\Omega)}, \alpha/\mu\}$$

for any $t > 0$.

Proof. We first consider the Model IPT1 i.e. w satisfies

$$w_t - d_w\Delta w + \mu w = \alpha v F(u), \quad x \in \Omega,\ \ t > 0. \tag{4.7}$$

Thanks to (2.16) we have $F(u) \in [0, F_m]$, (4.4), Lemma 4.1 and maximum principle it results

$$0 \le w \le \frac{1}{\mu}\max\{\|w_0\|_{L^\infty(\Omega)}, \alpha F_m\|v\|_{L^\infty(\Omega)}\}.$$

The upper bound of v obtained in the previous lemma ends the proof for Model IPT1. For second model, maximum principle gives

$$0 \le w \le \frac{1}{\mu}\max\{\|w_0\|_{L^\infty(\Omega)}, \alpha\|v\|_{L^\infty(\Omega)}\}.$$

Lemma 4.2 ends the proof. □

Lemma 4.3. *Under assumptions of Theorem* 4.1 *and* (4.4), *we have that*:

- *the solution to Model* IPT1 *satisfies*

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla w|^2 + d_w\int_\Omega |\Delta w|^2 + \frac{\mu}{2}\int_\Omega |\nabla w|^2 \le \frac{\alpha^2 F_m^2}{\mu}\int_\Omega \frac{|\nabla v|^2}{v} + \frac{\alpha^2 F_m^2}{\mu}\int_\Omega \frac{|\nabla u|^2}{(1+u)^4}, \tag{4.8}$$

- *the solution to Model* IPT2 *satisfies*

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla w|^2 + d_w\int_\Omega |\Delta w|^2 + \frac{\mu}{2}\int_\Omega |\nabla w|^2 \le \frac{\alpha^2}{2\mu}\int_\Omega \frac{|\nabla v|^2}{v}. \tag{4.9}$$

Proof. We first consider the Model IPT1 and multiply Eq. (1.3) by $-\Delta w$ and integrate by parts over Ω to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla w|^2 + d_w\int_\Omega |\Delta w|^2 + \mu\int_\Omega |\nabla w|^2 = -\alpha\int_\Omega vF(u)\Delta w.$$

Since

$$-\alpha\int_\Omega vF(u)\Delta w = \alpha\int_\Omega F(u)\nabla v\nabla w + \alpha F_m\int_\Omega \frac{v}{(1+u)^2}\nabla u\nabla w$$

and $|F(u)| \le F_m$, $|v| \le 1$ we have, thanks to the Young inequality

$$-\alpha\int_\Omega vF(u)\Delta w \le \frac{\alpha^2F_m^2}{\mu}\int_\Omega|\nabla v|^2 + \frac{\alpha^2F_m^2}{\mu}\int_\Omega\frac{|\nabla u|^2}{(1+u)^4} + \frac{\mu}{2}\int_\Omega|\nabla w|^2.$$

We now combine both identities to obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega|\nabla w|^2 + d_w\int_\Omega|\Delta w|^2 + \frac{\mu}{2}\int_\Omega|\nabla w|^2&\\ \le \frac{\alpha^2F_m^2}{\mu}\int_\Omega|\nabla v|^2 + \frac{\alpha^2F_m^2}{\mu}\int_\Omega\frac{|\nabla u|^2}{(1+u)^4}&.\end{aligned}$$

Since $1/v \ge 1$, the proof of (4.8) is completed.

To prove (4.9) we proceed in the same way. We multiply Eq. (1.9) by $-\Delta w$ and integrate by parts over Ω and right-hand side term satisfies

$$-\alpha\int_\Omega v\Delta w = \alpha\int_\Omega\nabla v\nabla w \le \frac{\alpha^2}{2\mu}\int_\Omega|\nabla v|^2 + \frac{\mu}{2}\int_\Omega|\nabla w|^2.$$

Since $1/v \ge 1$ we have

$$\int_\Omega|\nabla v|^2 \le \int_\Omega\frac{|\nabla v|^2}{v}$$

and

$$-\alpha\int_\Omega v\Delta w \le \frac{\alpha^2}{2\mu}\int_\Omega\frac{|\nabla v|^2}{v} + \frac{\mu}{2}\int_\Omega|\nabla w|^2.$$

The rest of the terms are treated as in the first case. □

Lemma 4.4. *Under assumptions of Theorem* 4.1 *we have that*

$$\frac{d}{dt}\int_\Omega\left|\frac{\nabla v}{v}\right|^2 \le -\lambda\int_\Omega\frac{|\nabla v|^2}{v} + \frac{F_m^2}{\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}\int_\Omega\frac{|\nabla u|^2}{(1+u)^4}. \tag{4.10}$$

Proof. We divide Eq. (1.4) by v and derivate the resulting equation to obtain

$$\frac{d}{dt}\frac{\nabla v}{v} = -\lambda\nabla v - \frac{F_m}{(1+u)^2}\nabla u.$$

Multiplying by $\nabla v/v$ and after integration over Ω we get

$$\frac{d}{dt}\frac{1}{2}\int_\Omega\left|\frac{\nabla v}{v}\right|^2 = -\lambda\int_\Omega\frac{|\nabla v|^2}{v} - F_m\int_\Omega\frac{\nabla u\nabla v}{v(1+u)^2}.$$

Since $1/(u+1) < 1$ and $v \ge \min\{\underline{v}_0,(\lambda-F_m)/\lambda\}$, by Young inequality

$$\begin{aligned}-F_m\frac{\nabla u\nabla v}{v(1+u)^2} &\le \frac{F_m^2}{2\lambda}\frac{|\nabla u|^2}{v(1+u)^4} + \frac{\lambda}{2}\frac{|\nabla v|^2}{v}\\ &\le \frac{F_m^2}{2\lambda\min\{\underline{v}_0,(\lambda-F_m)/\lambda\}}\frac{|\nabla u|^2}{(1+u)^4} + \frac{\lambda}{2}\frac{|\nabla v|^2}{v}\end{aligned}$$

which implies

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \left|\frac{\nabla v}{v}\right|^2 \le -\frac{\lambda}{2}\int_\Omega \frac{|\nabla v|^2}{v} + \frac{F_m^2}{2\lambda \min\{\underline{v}_0, (\lambda - F_m)/\lambda\}}\int_\Omega \frac{|\nabla u|^2}{(1+u)^4}.$$

By using $\lambda \min\{\underline{v}_0, (\lambda - F_m)/\lambda\} = \min\{\lambda\underline{v}_0, (\lambda - F_m)\}$ and multiplying by 2, we end the proof. □

Lemma 4.5. *Under assumptions of Theorem* 4.1, *for* $N \le 2$ *we have*

$$\frac{d}{dt}\int_\Omega (u+1)(\ln(u+1)-1) + \frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{u+1} \le \chi^2 c^* \langle u_0\rangle \|\Delta w\|^2_{L^2(\Omega)} + \chi^2(\langle u_0\rangle + 2)\int_\Omega |\nabla w|^2, \tag{4.11}$$

where c^* *is defined in* (4.21).

Proof. We multiply Eq. (1.2) by $\ln(u+1)$ and integrate over Ω to obtain

$$\frac{d}{dt}\int_\Omega (u+1)(\ln(u+1)-1) + \int_\Omega \frac{|\nabla u|^2}{u+1} = \chi\int_\Omega \frac{u}{u+1}\nabla u \nabla w. \tag{4.12}$$

We consider the right-hand side term, after integration by parts and thanks to non-negativity of u and the Young inequality we have

$$\begin{aligned}\chi\int_\Omega \frac{u}{u+1}\nabla u\nabla w &= \chi\int_\Omega \nabla(u - \ln(u+1))\nabla w\\ &= -\chi\int_\Omega (u - \ln(u+1))\Delta w\\ &= -\chi\int_\Omega u\Delta w - \chi\int_\Omega \frac{\nabla u\nabla w}{u+1}\\ &\le -\chi\int_\Omega u\Delta w + \chi^2\int_\Omega |\nabla w|^2 + \frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{u+1}.\end{aligned} \tag{4.13}$$

In order to bound the term $u\Delta w$ in an appropriate norm we make the following decomposition

$$\begin{aligned}u &= (u+1)^{\frac12}(u+1)^{\frac12} - 1\\ &= \left((u+1)^{\frac12} - \left\langle (u+1)^{\frac12}\right\rangle\right)^2 + 2(u+1)^{\frac12}\left\langle (u+1)^{\frac12}\right\rangle - \left\langle (u+1)^{\frac12}\right\rangle^2 - 1.\end{aligned}$$

Since $-\left\langle (u+1)^{\frac12}\right\rangle^2 - 1$ is independent of x we have

$$-\int_\Omega \left(\left\langle (u+1)^{\frac12}\right\rangle^2 + 1\right)\Delta w = 0$$

and therefore

$$\begin{aligned}-\chi\int_\Omega u\Delta w = -\chi\int_\Omega &\left((u+1)^{\frac12} - \left\langle (u+1)^{\frac12}\right\rangle\right)^2\Delta w\\ &- 2\chi\left\langle (u+1)^{\frac12}\right\rangle\int_\Omega (u+1)^{\frac12}\Delta w.\end{aligned}$$

Notice that

$$\left\langle (u+1)^{\frac{1}{2}} \right\rangle \le \langle (u+1)\rangle^{\frac{1}{2}} \le \langle u+1\rangle^{\frac{1}{2}} \le (\langle u_0\rangle + 1)^{\frac{1}{2}}$$

and after integration by parts in the last term it follows

$$\begin{aligned} 2\chi \left\langle (u+1)^{\frac{1}{2}} \right\rangle \left| \int_\Omega (u+1)^{\frac{1}{2}} \Delta w \right| &\le \chi(\langle u_0\rangle + 1)^{\frac{1}{2}} \left| \int_\Omega \frac{\nabla u}{(u+1)^{\frac{1}{2}}} \nabla w \right| \\ &\le \frac{1}{4} \int_\Omega \frac{|\nabla u|^2}{u+1} + \chi^2(\langle u_0\rangle + 1) \int_\Omega |\nabla w|^2. \end{aligned} \tag{4.14}$$

Then

$$\begin{aligned} -\chi \int_\Omega u \Delta w \le{}& -\chi \int_\Omega \left((u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right)^2 \Delta w \\ &+ \frac{1}{4} \int_\Omega \frac{|\nabla u|^2}{u+1} + \chi^2(\langle u_0\rangle + 1) \int_\Omega |\nabla w|^2. \end{aligned} \tag{4.15}$$

The other term on the right-hand side is treated below

$$\begin{aligned} &-\chi \int_\Omega \left((u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right)^2 \Delta w \\ &\quad \le \frac{1}{4\langle u_0\rangle c^*} \left\| (u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right\|_{L^4(\Omega)}^4 + \chi^2 c^* \langle u_0\rangle \|\Delta w\|_{L^2(\Omega)}^2, \end{aligned} \tag{4.16}$$

where c^* is a positive constant defined in (4.21).

Since $N \le 2$ we have that, thanks to the Gagliardo–Nirenberg inequality we know that for any $z \in H^1(\Omega)$:

$$\|z\|_{L^4(\Omega)} \le C_{GN} \|z\|_{L^2(\Omega)}^{\frac{1}{2}} \|z\|_{W^{1,N}(\Omega)}^{\frac{1}{2}} \le C_{GN} |\Omega|^{\frac{2-N}{4}} \|z\|_{L^2(\Omega)}^{\frac{1}{2}} \|z\|_{H^1(\Omega)}^{\frac{1}{2}} \tag{4.17}$$

and by Poincaré inequality, we have that

$$\|z - \langle z\rangle\|_{H^1(\Omega)}^2 \le C_P \|\nabla z\|_{L^2(\Omega)}^2. \tag{4.18}$$

We apply (4.17) to the term $(u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle$ to obtain

$$\begin{aligned} &\left\| (u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right\|_{L^4(\Omega)}^4 \\ &\quad \le C_{GN}^4 |\Omega|^{2-N} \left\| (u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right\|_{L^2(\Omega)}^2 \left\| (u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right\|_{H^1(\Omega)}^2. \end{aligned}$$

Note that

$$\left\| (u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right\|_{L^2(\Omega)}^2 = \int_\Omega (u+1) - \left\langle (u+1)^{\frac{1}{2}} \right\rangle^2 \le |\Omega| \langle u\rangle$$

and thanks to (4.18) it follows

$$\left\| (u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle \right\|_{H^1(\Omega)}^2 \le C_P \left\| \nabla (u+1)^{\frac{1}{2}} \right\|_{L^2(\Omega)}^2 = \frac{C_P}{4} \int_\Omega \frac{|\nabla u|^2}{u+1} \tag{4.19}$$

and it entails

$$\left\|(u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle\right\|_{L^4(\Omega)}^4 \leq \frac{C_{GN}^4 C_P |\Omega|^{3-N} \langle u_0 \rangle}{4} \int_\Omega \frac{|\nabla u|^2}{u+1}. \tag{4.20}$$

Then, we define the positive constant c^*:

$$c^* := \frac{C_{GN}^4 C_P |\Omega|^{3-N}}{4}, \tag{4.21}$$

where C_{GN} is the constant given by the Gagliardo–Nirenberg inequality and C_P is the constant given in (4.18).

Next we use (4.20) in (4.16) to obtain

$$-\chi \int_\Omega \left((u+1)^{\frac{1}{2}} - \left\langle (u+1)^{\frac{1}{2}} \right\rangle\right)^2 \Delta w \leq \frac{1}{4} \int_\Omega \frac{|\nabla u|^2}{u+1} + \chi^2 c^* \langle u_0 \rangle \|\Delta w\|_{L^2(\Omega)}^2. \tag{4.22}$$

Thanks to (4.22) and (4.15) we obtain

$$-\chi \int_\Omega u \Delta w \leq \frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{u+1} + \chi^2 c^* \langle u_0 \rangle \|\Delta w\|_{L^2(\Omega)}^2 + \chi^2 (\langle u_0 \rangle + 1) \int_\Omega |\nabla w|^2, \tag{4.23}$$

which ends the proof by substitution into (4.13). □

Lemma 4.6. *Under assumptions of Theorem* 4.1 *we have that*

$$\int_\Omega u(\ln u - 1) + \int_\Omega |\nabla w|^2 + \int_\Omega \left|\frac{\nabla v}{v}\right|^2 + \int_0^\infty \int_\Omega |\nabla w|^2 + \int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u+1} \leq C.$$

Proof. We first consider IPT1 model. We define the constant A satisfying:

$$A > \max\left\{\frac{\chi^2 c^* \langle u_0 \rangle}{2 d_w}, \frac{\chi^2(\langle u_0 \rangle + 2)}{\mu}\right\},$$

$$A \frac{2\alpha^2 F_m}{\mu}\left(1 + \frac{F_m^2}{\lambda \min\{\lambda \bar{v}_0, (\lambda - F_m)\}}\right) < \frac{1}{4}.$$

Notice that under assumption (4.5) it is possible to find such A satisfying the previous inequalities. Let B defined by

$$B := \frac{2\alpha^2 F_m^2}{\mu\lambda} A$$

which implies

$$\chi^2 c^* \langle u_0 \rangle - 2 d_w A < 0, \quad \chi^2(\langle u_0 \rangle + 2) - A\mu < 0, \tag{4.24}$$

$$\frac{2\alpha^2 F_m^2 A}{\mu} - \lambda B = 0. \tag{4.25}$$

We multiply (4.8) by $2A$, (4.10) by B and we add both expressions to (4.11) to obtain

$$\frac{d}{dt}\left[\int_\Omega u(\ln u - 1) + A \int_\Omega |\nabla w|^2 + B \int_\Omega \left|\frac{\nabla v}{v}\right|^2\right] \leq -\frac{1}{4} \int_\Omega \frac{|\nabla u|^2}{u+1}$$
$$+ (\chi^2 c^* \langle u_0 \rangle - 2 d_w A) \int_\Omega |\Delta w|^2 + (\chi^2(\langle u_0 \rangle + 2) - A\mu) \int_\Omega |\nabla w|^2$$

$$+\left(\frac{2\alpha^2F_m^2A}{\mu}-\lambda B\right)\int_\Omega\frac{|\nabla v|^2}{v}+\left(\frac{2\alpha^2F_m^2A}{\mu}+\frac{BF_m^2}{\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}\right)$$
$$\times\int_\Omega\frac{|\nabla u|^2}{(1+u)^4}.$$

Since

$$\frac{2\alpha^2F_m^2A}{\mu}+\frac{BF_m^2}{\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}=A\frac{2\alpha^2F_m^2}{\mu}\left(1+\frac{F_m^2}{\lambda\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}\right)$$

and

$$\max\left\{\frac{\chi^2c^*\langle u_0\rangle}{2d_w},\frac{\chi^2(\langle u_0\rangle+2)}{\mu}\right\}\frac{8\alpha^2F_m^2}{\mu}\left(1+\frac{F_m^2}{\lambda\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}\right)<1,$$

we have that there exists $\varepsilon>0$ such that

$$A\frac{2\alpha^2F_m^2}{\mu}\left(1+\frac{F_m^2}{\lambda\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}\right)-\frac{1}{4}<-\varepsilon.$$

After integration, thanks to (4.24) and (4.25) we have that

$$\int_\Omega u(\ln u-1)+\int_\Omega|\nabla w|^2+\int_\Omega\left|\frac{\nabla v}{v}\right|^2+\int_0^\infty\int_\Omega|\nabla w|^2+\int_0^\infty\int_\Omega\frac{|\nabla u|^2}{u+1}\le C.$$

For Model IPT2 we proceed in the same way, with the following constants:

$$A>\max\left\{\frac{\chi^2c^*\langle u_0\rangle}{2d_w},\frac{\chi^2(\langle u_0\rangle+2)}{\mu}\right\},$$
$$A\frac{8\alpha^2}{\mu\lambda}\frac{F_m^2}{\min\{\lambda\underline{v}_0,(\lambda-F_m)\}}<1$$

and

$$B:=\frac{\alpha^2}{\mu\lambda}A.$$

Notice that under assumption (4.6) it is possible to find such A satisfying the previous inequalities. □

Corollary 4.1. *Thanks to Lemma* 4.6, *integrating in* (4.10) *and* (4.8) *we have*:

$$\int_0^\infty\int_\Omega\frac{|\nabla v|^2}{v}\le C, \tag{4.26}$$

$$\int_\Omega|\nabla w|^2+\int_0^\infty\int_\Omega|\Delta w|^2\le C. \tag{4.27}$$

It follows by Lemma 4.1 *that* $v\ge 1-\frac{F_m}{\lambda}$ *and then*

$$\int_0^\infty\int_\Omega|\nabla v|^2\le C. \tag{4.28}$$

Lemma 4.7. *Under assumptions of Theorem* 4.1, *for* $N \leq 2$ *we have that*

$$\int_0^\infty \int_\Omega |\nabla w_t|^2 + \int_\Omega |\Delta w|^2 \leq C. \tag{4.29}$$

Proof. On multiplying Eq. (1.3) by $-\Delta w_t$ and integrating by parts we obtain

$$\int_\Omega |\nabla w_t|^2 + \frac{d}{dt}\frac{d_w}{2}\int_\Omega |\Delta w|^2 + \frac{d}{dt}\frac{\mu}{2}\int_\Omega |\nabla w|^2$$
$$= \alpha \int_\Omega F(u)\nabla v \nabla w_t + \int_\Omega \frac{\alpha F_m v}{(u+1)^2}\nabla u \nabla w_t.$$

As in Lemma 4.3 we find the following bound

$$\alpha \int_\Omega F(u)\nabla v \nabla w_t \leq \alpha^2 F_m^2 \int_\Omega \frac{|\nabla v|^2}{v} + \frac{1}{4}\int_\Omega |\nabla w_t|^2$$

and

$$\int_\Omega \frac{\alpha F_m v}{(u+1)^2}\nabla u \nabla w_t \leq \alpha^2 F_m^2 \int_\Omega \frac{|\nabla u|^2}{(1+u)^2} + \frac{1}{4}\int_\Omega |\nabla w_t|^2.$$

Hence,

$$\frac{1}{2}\int_\Omega |\nabla w_t|^2 + \frac{d}{dt}\frac{d_w}{2}\int_\Omega |\Delta w|^2 + \frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla w|^2$$
$$\leq \alpha^2 F_m^2 \int_\Omega \frac{|\nabla v|^2}{v} + \alpha^2 F_m^2 \int_\Omega \frac{|\nabla u|^2}{(1+u)^2}.$$

After integration, the result is a consequence of Lemma 4.6, (2.16) and (4.26). □

Lemma 4.8. *Under assumptions of Theorem* 4.1 *we have that*

$$\int_\Omega u^2 + \int_0^\infty \int_\Omega |\nabla u|^2 \leq c.$$

Proof. We first notice that

$$\int_\Omega u^2 = \int_\Omega |u - \langle u \rangle|^2 + \frac{1}{|\Omega|}\left|\int_\Omega u\right|^2.$$

We multiply Eq. (1.2) by u and integrate over Ω to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 = \chi \int_\Omega u \nabla u \nabla w. \tag{4.30}$$

Taking into account

$$\chi \int_\Omega u\nabla u \nabla w = \chi \int_\Omega (u - \langle u_0 \rangle)\nabla u \nabla w + \chi \langle u_0 \rangle \int_\Omega \nabla u \nabla w$$

with

$$\chi \langle u_0 \rangle \int_\Omega \nabla u \nabla w \leq \frac{1}{4}\int_\Omega |\nabla u|^2 + \chi^2 \langle u_0 \rangle^2 \int_\Omega |\nabla w|^2$$

and using the Young inequality we arrive at

$$\chi \int_\Omega (u - \langle u_0 \rangle)\nabla u \nabla w \le \frac{1}{4}\int_\Omega |\nabla u|^2 + \chi^2 \int_\Omega (u - \langle u_0\rangle)^2 |\nabla w|^2$$
$$\le \frac{1}{4}\int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega (u - \langle u_0\rangle)^3 + C(\varepsilon)\int_\Omega |\nabla w|^6.$$

By the Gagliardo–Nirenberg inequality we have that for any $z \in H^1(\Omega)$:

$$\|z\|_{L^3(\Omega)} \le C_{\frac{1}{3}} \|z\|^{\frac{1}{3}}_{L^1(\Omega)} \|z\|^{\frac{2}{3}}_{W^{1,N}(\Omega)}.$$

Setting $z = u - \langle u_0\rangle$ we obtain

$$\int_\Omega (u - \langle u_0\rangle)^3 \le C^3_{\frac{1}{3}} \langle u_0\rangle C(\Omega)\left(\int_\Omega |\nabla u|^N\right)^{\frac{2}{N}} \le C(\Omega, \langle u_0\rangle, C_{\frac{1}{3}}, N)\int_\Omega |\nabla u|^2.$$

By Lemmas 4.6 and 4.7 we know that $\|\Delta w\|_{L^2(\Omega)}$ and $\|\nabla w\|_{L^2(\Omega)}$ are uniformly bounded in time, hence using the Sobolev embedding we find

$$\int_\Omega |\nabla w|^6 \le C(\Omega)\left(\int_\Omega |\Delta w|^2 + \int_\Omega |\nabla w|^2\right)^3 \le c\left(\int_\Omega |\Delta w|^2 + \int_\Omega |\nabla w|^2\right).$$

Taking ε small enough, we get

$$\chi \int_\Omega u \nabla u \nabla w \le \frac{1}{2}\int_\Omega |\nabla u|^2 + c'\left(\int_\Omega |\Delta w|^2 + \int_\Omega |\nabla w|^2\right). \tag{4.31}$$

Using the previous inequality in (4.30) we obtain

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \le \frac{1}{2}\int_\Omega |\nabla u|^2 + c'\left(\int_\Omega |\Delta w|^2 + \int_\Omega |\nabla w|^2\right). \tag{4.32}$$

It remains to integrate the last inequality over $(0, \infty)$ and apply Lemma 4.6 and (4.27) to complete the proof. □

Lemma 4.9. *Under assumptions of Theorem* 4.1 *we have that,*

$$\int_\Omega |u - \langle u_0\rangle|^2 \to 0 \quad \text{as } t \to \infty.$$

Proof. We consider a function $k : [0, \infty) \to \mathbb{R}_+$ such that:

(i) $k \ge 0$,
(ii) $\int_0^\infty k = c_0 < \infty$,

and either (iii$_1$) or (iii$_2$):

(iii$_1$) $|k'| \le c_1 < \infty$,
(iii$_2$) $|k(t+s) - k(t)| \le \epsilon(t)$ for all $s > 0$, where $\epsilon(t) \to 0$ as $t \to \infty$,

then, thanks to Lemma 5.1 in Ref. 10, we obtain that

$$k(t) \to 0 \quad \text{as } t \to \infty.$$

We define the function

$$k_1(t) := \int_\Omega (u - \langle u_0 \rangle)^2.$$

Then, thanks to Lemma 4.8 and the Poincaré inequality k_1 satisfies (ii). To prove (iii_2) we proceed as follows:

$$\frac{d}{dt} k_1(t) = 2 \int_\Omega u_t (u - \langle u_0 \rangle) = 2 \int_\Omega u u_t = \frac{d}{dt} \int_\Omega u^2.$$

By (4.32) we have that

$$\frac{d}{dt} \int_\Omega u^2 \le -\frac{1}{2} \int_\Omega |\nabla u|^2 + c' \left(\int_\Omega |\Delta w|^2 + \int_\Omega |\nabla w|^2 \right),$$

integrating over $(t, t+s)$ we obtain

$$\int_t^{t+s} \left[\frac{d}{dt} \int_\Omega u^2 \right] = \int_\Omega u^2(t+s) - \int_\Omega u^2(t)$$

and then

$$\int_\Omega u^2(t+s) - \int_\Omega u^2(t) \le -\frac{1}{2} \int_t^{t+s} \int_\Omega |\nabla u|^2 + c' \int_t^{t+s} \int_\Omega |\Delta w|^2 + c' \int_t^{t+s} \int_\Omega |\nabla w|^2.$$

In the same way as (4.32) we may obtain the inequality

$$\frac{d}{dt} \int_\Omega u^2 \ge -\frac{1}{2} \int_\Omega |\nabla u|^2 - c' \left(\int_\Omega |\Delta w|^2 + \int_\Omega |\nabla w|^2 \right)$$

which give us, after integration

$$\left| \int_\Omega u^2(t+s) - \int_\Omega u^2(t) \right| \le \int_t^{t+s} \int_\Omega |\nabla u|^2 + c' \int_t^{t+s} \int_\Omega |\Delta w|^2 + c' \int_t^{t+s} \int_\Omega |\nabla w|^2.$$

Now we may set

$$\epsilon(t) := \int_t^\infty \int_\Omega \left(|\nabla u|^2 + c' \int_\Omega |\Delta w|^2 + c' \int_\Omega |\nabla w|^2 \right).$$

In light of Lemmas 4.6–4.8 we have that $\epsilon(t) \to 0$ as $t \to \infty$ then, k_1 satisfies (iii_2). Due to the positivity of k_1 and the Cauchy–Schwartz inequality we may apply Ref. 10 to end the proof. □

Corollary 4.2. *Under assumptions of Theorem* 4.1 *we have that, for any* $p < \infty$:

$$\|u - \langle u_0 \rangle\|_{L^p(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Proof. We notice that, for any $z \in L^\infty(\Omega)$ we have that

$$\int_\Omega z^p \leq C\|z\|_{L^2(\Omega)}\|z\|^{p-1}_{L_\infty(\Omega)},$$

as a consequence of the previous lemma we prove the convergence for any $p < \infty$. □

Lemma 4.10. *Under assumptions of Theorem* 4.1 *we have that, for any* $p \in [1, \infty)$:

$$\int_\Omega |w(t) - \langle w(t) \rangle|^p \to 0 \quad \text{as } t \to \infty.$$

Proof. As in the previous lemma we consider the function

$$k_2(t) := \int_\Omega |w(t) - \langle w(t) \rangle|^p.$$

Notice that $k_2 \geq 0$ and thanks to the Poincaré inequality and Lemma 4.6:

$$\int_0^\infty k_2(t) \leq C_p\|w\|^{p-2}_{L^\infty(\Omega)} \int_0^\infty \int_\Omega |\nabla w|^2 \leq C.$$

Since

$$\frac{d}{dt}k_2(t) = p\int_\Omega (w_t - \langle w_t \rangle)|w - \langle w \rangle|^{p-2}(w - \langle w \rangle)$$

and thanks to Lemma 4.2 and boundedness of w we obtain the upper bound for $|k'|$ in terms of $\|w_t\|_{L^1(\Omega)}$, i.e.

$$\left|\frac{d}{dt}k_2(t)\right| \leq c\int_\Omega |w_t| \leq C_\alpha + d_w \int_\Omega |\Delta w|,$$

where C_α is some constant depending on F_m, α and $\sup_{t>0} \|w(t)\|_{L^\infty(\Omega)}$. Finally note that k_2 satisfies (iii$_1$) because

$$\int_\Omega |\Delta w| \leq \frac{1}{2} + \frac{1}{2}\int_\Omega |\Delta w|^2 \leq C < \infty$$

which ends the proof. □

In order to finish the proof of Theorem 4.1, we have to obtain the convergence of $\int_\Omega w dx$. We first study the convergence of the function $F(u)$.

Lemma 4.11.

$$\int_0^\infty \int_\Omega |F(u) - F(\langle u_0 \rangle)|^2 \leq C < \infty.$$

Proof. The proof is a consequence of the mean value theorem, Poincaré inequality and Lemma 4.8:

$$\int_0^\infty \int_\Omega |F(u) - F(\langle u_0 \rangle)|^2 \le \|F'\|^2_{L^\infty} \int_0^\infty \int_\Omega |u - \langle u_0 \rangle|^2$$
$$\le \|F'\|^2_{L^\infty} \int_0^\infty \int_\Omega |\nabla u|^2 \le C < \infty. \qquad \square$$

Lemma 4.12. *Under assumptions of Theorem* 4.1 *we have that, for any* $p \in [1, \infty)$:

$$\int_\Omega \left| v - 1 + \frac{1}{\lambda} F(\langle u_0 \rangle) \right|^p \to 0 \quad \textit{as } t \to \infty.$$

Proof. Let y be the solution to the logistic equation

$$y' = \lambda y \left(1 - \frac{1}{\lambda} F(\langle u_0 \rangle) - y \right).$$

Thanks to assumption (4.1) we have that

$$1 - \frac{1}{\lambda} F(\langle u_0 \rangle) > 0$$

and for $y(0) > 0$, it is well known that

$$y(t) \to 1 - \frac{1}{\lambda} F(\langle u_0 \rangle) \text{ exponentially,} \quad \text{as } t \to +\infty. \tag{4.33}$$

We now consider Eq. (1.4):

$$\frac{d}{dt} v = \lambda v \left(1 - v - \frac{1}{\lambda} F(u)\right).$$

Then

$$\frac{d}{dt}(v - y) = (v - y)(\lambda - \lambda(v + y)) - vF(u) + yF(\langle u_0 \rangle)$$
$$= (v - y)(\lambda - \lambda(v + y) - F(\langle u_0 \rangle)) - v(F(u) - F(\langle u_0 \rangle)).$$

Notice that $F(u) \le F_m$ for any $u \ge 0$ and therefore $\lambda - F(u) \ge \lambda - F_m$. In view of (1.4) we claim that $v(t) \ge 1 - F_m/\lambda$ for t large enough. Since $y_0 > 0$ we have that for T large enough

$$\left| y - 1 + \frac{1}{\lambda} F(\langle u_0 \rangle) \right| \le \frac{1}{2} - \frac{F_m}{2\lambda}, \quad \text{for } t > T.$$

It follows that

$$\frac{d}{dt}(v - y) + \frac{\lambda - F_m}{2}(v - y) \le -v(F(u) - F(\langle u_0 \rangle)).$$

On multiplying by $|v - y|^{p-2}(v - y)$ (for $p \ge 2$) and integrating over Ω, thanks to the Young inequality and boundedness of v and F, we have that

$$\frac{d}{dt} \frac{1}{p} \int_\Omega |v - y|^p + C_1 \int_\Omega |v - y|^p \le C_2 \int_\Omega |F(u) - F(\langle u_0 \rangle)|^p \quad \text{for } t > T.$$

Since F is bounded and $p \geq 2$:

$$\int_\Omega |F(u) - F(\langle u_0 \rangle)|^p \leq C \int_\Omega |F(u) - F(\langle u_0 \rangle)|^2,$$

then, after integration over (T, ∞) and thanks to Lemma 4.11 we obtain

$$\int_T^\infty \int_\Omega |v - y|^p \leq C_3 \int_T^\infty \int_\Omega |F(u) - F(\langle u_0 \rangle)|^2 \leq C$$

and therefore

$$\int_0^\infty \int_\Omega |v - y|^p = \int_0^T \int_\Omega |v - y|^p + \int_T^\infty \int_\Omega |v - y|^p \leq C. \tag{4.34}$$

As in the previous lemma we consider the function

$$k_3(s) := \int_\Omega |v - y|^p$$

which clearly satisfies (i) and (ii). (iii_1) is a consequence of boundedness of v, F and y'. We now apply Lemma 5.1 in Ref. 10, to prove that

$$\int_\Omega |v - y|^p \to 0, \quad \text{as } t \to \infty,$$

(4.33) and the above inequality prove the lemma. □

Lemma 4.13. *Let $\bar{w}$ as defined in Theorem* 4.1. *Then,*

$$|\langle w(t) \rangle - \bar{w}|^p \to 0, \quad \text{as } t \to \infty.$$

Proof. We first consider the IPT1 problem, and we integrate (1.3) over Ω to obtain:

$$\frac{d}{dt} \int_\Omega w + \mu \int_\Omega w = \alpha \int_\Omega vF(u).$$

Then

$$\frac{d}{dt} \int_\Omega (w - \bar{w}) + \mu \int_\Omega (w - \bar{w}) = \int_\Omega (\alpha vF(u) - \mu\bar{w}). \tag{4.35}$$

We add and subtract to the last term in the previous equation the term $\alpha\bar{v} \int_\Omega F(u)$ and replace $\bar{w}$ by its definition to obtain

$$\int_\Omega (\alpha vF(u) - \mu\bar{w}) = \int_\Omega (v - \bar{v})\alpha F(u) + \frac{\alpha}{|\Omega|} \int_\Omega v \int_\Omega (F(u) - F(\langle u_0 \rangle))$$
$$+ \alpha F(\langle u_0 \rangle) \int_\Omega \left(v - 1 + \frac{1}{\lambda} F(\langle u_0 \rangle) \right)$$

and by boundedness of v and F we obtain for any $p \geq 2$:

$$\left| \int_\Omega (\alpha vF(u) - \mu\bar{w}) \right|^p \leq C_1 \int_\Omega |v - \bar{v}|^p + C_2 \int_\Omega |F(u) - F(\langle u_0 \rangle)|^2$$
$$+ C_3 \int_\Omega \left| v - 1 + \frac{1}{\lambda} F(\langle u_0 \rangle) \right|^p.$$

Integrating over $(0,\infty)$ and making use of Lemmata 4.11 and 4.12 and (4.28) we conclude

$$\int_0^\infty \left|\int_\Omega (\alpha v F(u) - \mu\bar{w})\right|^p \le C. \tag{4.36}$$

On multiplying in (4.35) by

$$\left|\int_\Omega (w-\bar{w})\right|^{p-2}\int_\Omega (w-\bar{w}),$$

we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{p}\left|\int_\Omega (w-\bar{w})\right|^p &+ \mu\left|\int_\Omega (w-\bar{w})\right|^p \\ &= \left|\int_\Omega (w-\bar{w})\right|^{p-2}\int_\Omega (w-\bar{w})\int_\Omega (\alpha v F(u) - \mu\bar{w}). \end{aligned}\tag{4.37}$$

Because of uniform boundedness of w, v and F we infer

$$\left|\frac{d}{dt}\frac{1}{p}\left|\int_\Omega (w-\bar{w})\right|^p\right| \le C. \tag{4.38}$$

By the Young inequality and (4.37) we obtain

$$\frac{d}{dt}\frac{1}{p}\left|\int_\Omega (w-\bar{w})\right|^p + c\left|\int_\Omega (w-\bar{w})\right|^p \le c\int_\Omega |\alpha v F(u) - \mu\bar{w}|^p$$

integrating over $(0,\infty)$ we obtain, by (4.36):

$$\int_0^\infty \left|\int_\Omega (w-\bar{w})\right|^p \le C. \tag{4.39}$$

Equations (4.38), (4.37) and Lemma 5.1 in Ref. 10, prove the result for IPT1 model.

ITP2 model is treated in the same way, after integration we obtain

$$\frac{d}{dt}\int_\Omega w + \mu\int_\Omega (w-\bar{w}) = \int_\Omega (\alpha v - \mu\bar{w}) = \alpha\int_\Omega (v-\bar{v}).$$

We multiply by $|\int_\Omega (w-\bar{w})|^{p-2}\int_\Omega (w-\bar{w})$ and obtain

$$\frac{d}{dt}\frac{1}{p}\left|\int_\Omega (w-\bar{w})\right|^p + \mu\left|\int_\Omega (w-\bar{w})\right|^p = \alpha\left|\int_\Omega (w-\bar{w})\right|^{p-2}\int_\Omega (w-\bar{w})\int_\Omega (v-\bar{v}).$$

The Young inequality provides

$$\frac{d}{dt}\frac{1}{p}\left|\int_\Omega (w-\bar{w})\right|^p + \frac{\mu}{2}\left|\int_\Omega (w-\bar{w})\right|^p \le c\left|\int_\Omega (v-\bar{v})\right|^p. \tag{4.40}$$

Boundedness of v and w gives

$$\left|\frac{d}{dt}\frac{1}{p}\left|\int_\Omega (w-\bar{w})\right|^p\right| \le C.$$

Integrating in (4.40) over $(0,\infty)$ we obtain

$$\int_0^\infty \left|\int_\Omega (w-\bar{w})\right|^p \le C.$$

To end the proof we apply again Ref. 10, to the function

$$k(t) := \left|\int_\Omega (w-\bar{w})\right|^p. \qquad \square$$

Now we are in a position to complete the proof of Theorem 4.1. It is a consequence of Lemmata 4.9, 4.10, 4.12 and 4.13.

5. Conclusions and Discussion

We have studied two mathematical models which extend the simple logistic population model of prey population

$$v_t = \lambda v(1-v) - mv, \tag{5.1}$$

where m is the mortality rate which depends on the averaged density of predators $m = F(\bar{u})$. It is easily seen that for $\lambda > m$, Eq. (5.1) has a unique stable steady state $\bar{v} = 1 - \frac{m}{\lambda}$. We have considered weak feedback between predator and prey assuming that the consumption of prey affects only the spatial distribution of predator and its influence on birth rate of predators is negligible. Two possible strategies of search are used by the predator: random dispersal and taxis-oriented movement. In the simplest case, when the only strategy is the random dispersal, the situation is more similar to that described by the O.D.E. (5.1). Indeed, by setting $\chi = 0$ and removing the second column and second row in the matrix (3.9) we deduce that the space homogeneous steady state $\bar{v}$ is linearly stable without any additional assumptions on the parameters. We have shown that the steady state remains stable if we take into account an additional component of predator search strategy namely taxis toward some chemical released by prey — a smell of prey (Model IPT2). However, the spatially homogeneous steady state may become unstable if the chemoattractant is released by prey injured during capturing (Model IPT1) and $\Lambda = \frac{\chi\alpha\bar{u}}{\lambda}$ is big enough (Theorem 3.1). Thus, the steady state may be unstable if chemotactic sensitivity χ or the rate of release of the chemoattractant α is big enough. It is also worth noticing that in the case of very big initial averaged density of predator, we have that the spatially homogeneous steady state may be unstable even if χ and α are at relatively low level. This effect may be interpreted as the result of interference competition among predators since the total rate of consumption of prey by predators cannot exceed some threshold value even if the density level of predator is huge.

The instability result seems to be counterintuitive in the light of stability results known for full predator–prey system of Rosenzweig–MacArthur type with prey-taxis. For the latter case it was shown in Refs. 21 and 1 that prey-taxis enhances the stability of the spatially homogeneous steady state which is stable also in the case of O.D.E. system. The effect of instability in Model IPT1 reflects a positive feedback between the density of predators and production of chemoattractant — the effect which resembles the classical Keller–Segel model of chemotaxis. Notice also that for both models studied in this paper, the convergence of solutions to the constant steady state holds (see Sec. 4) provided some restrictions on parameters and the size of averaged initial density of predators and prey are satisfied.

In the case of Model IPT1 when the chemical emitted by prey is due to damage of prey body during capturing, we have proved existence of inhomogeneous steady states. One of defend strategies of prey against predator is aggregation, which decrease the predation risk per prey. Our results suggest that even in the case of models with weak coupling between predator and prey, the formation of prey aggregates may result solely from specific prey–predator interactions provided the search strategy of predator admits tactic migration toward the increasing concentration of the chemical released by injured prey.

The stability of non-homogeneous steady states is the subject of further studies. The global dynamics of solutions to both models is surprisingly complicated, in particular it is unclear what could be the long-time asymptotics of solutions when the spatially homogeneous steady state is unstable. The compactness of bounded trajectories of the corresponding dynamical system is unclear due to the lack of elliptic operator in the last equation.

A natural extension of the models is to consider the dispersal of prey population as well as the impact of prey on the rate of growth of predator or assuming bounded functional response of predator of type Holling I or Holling II.

Acknowledgments

J.I.T. is supported by Spanish Ministry of Economy under Grant Number MTM2013-42907-P (MICINN). D.W. was supported by the Polish Ministry of Science and Higher Education under the Grant N201547638.

References

1. B. Aiseba, M. Bendahmane and A. Noussair, A reaction–diffusion system modeling predator–prey with prey-taxis, *Nonlinear Anal. Real World Anal.* **9** (2008) 2086–2105.
2. N. D. Alikakos, L^p bounds of solutions of reaction–diffusion equations, *Comm. Partial Differential Equations* **4** (1979) 827–868.
3. N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.* **25** (2015) 1663–1763, Doi: 10.1142/S021820251550044X.
4. A. Chakrabortya, M. Singh, M. D. Lucy and P. Ridland, Predator–prey model with prey-taxis and diffusion, *Math. Comput. Model.* **46** (2007) 482–498.
5. M. A. J. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: Dynamic heterogeneity, *Netw. Heterog. Med.* **1** (2006) 399–439.
6. J. W. Cholewa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems* (Cambridge Univ. Press, 2000).
7. M. Connover, *Predator–Prey Dynamics*: *The Role of Olfaction* (CRC Press, 2007).

8. C. Cosner, D. L. DeAngelis, J. S. Ault and D. B. Olson, Effects of spatial grouping on the functional response of predators, *Theor. Popul. Biol.* **56** (1999) 65–75.
9. M. C. Ferrari, B. D. Wisenden and D. P. Chivers, Chemical ecology of predator–prey interactions in aquatic ecosystems: A review and prospectus, *Canad. J. Zool.* **33** (2010) 698–724.
10. A. Friedman and J. I. Tello, Stability of solutions of chemotaxis equations in reinforced random walks, *J. Math. Anal. Appl.* **272** (2002) 138–163.
11. Z. M. Gliwicz and D. Wrzosek, Predation mediated coexistence of large and small bodied daphnia at different food levels, *Amer. Nat.* **172** (2008) 358–374.
12. D. Grunbaum, Using spatially explicit models to characterize foraging performance in heterogeneous landscapes, *Amer. Nat.* **151** (1998) 97–115.
13. M. R. Heithaus and J. J. Vando, Predator–prey interactions, in *Biology of Sharks and Their Relatives*, eds. J. C. Carrier and J. A. Musick, 2nd edn. (CRC Press, 2012), pp. 505–547.
14. D. Henry, *Geometric Theory of Semilinear Parabolic Equations* (Springer-Verlag, 1981).
15. T. Hillen and K. J. Painter, A user's guide to PDE models for chemotaxis, *J. Math. Biol.* **58** (2009) 183–217.
16. T. Hillen, K. J. Painter and M. Winkler, Convergence of a cancer invasion model to a logistic chemotaxis model, *Math. Models Methods Appl. Sci.* **23** (2013) 165–198.
17. C. D. Hoefler, M. Taylor and E. M. Jakob, Chemosensory response to prey in *Phidippus audax* and *Pardosa milvina*, *J. Arachnol.* **30** (2002) 155–158.
18. A. Jüngel, Diffusive and nondiffusive population models, in *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*, eds. G. Naldi, L. Pareschi and G. Toscani (Birkhäuser, 2010).
19. P. Kareiva and G. Odell, Swarms of predators exhibit "prey-taxis" if individual predators use area-restricted search, *Amer. Nat.* **130** (1987) 233–270.
20. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Vol. 23 (Amer. Math. Soc., 1968).
21. J. M. Lee, T. Hillen and M. A. Lewis, Pattern formation in prey-taxis systems, *J. Biol. Dynam.* **3** (2009) 551–573.
22. A. Leung, Limiting behavior for a prey–predator model with diffusion and crowding effects, *J. Math. Biol.* **6** (1978) 87–93.
23. O. M. Lonnstedt, M. I. McCormick and D. P. Chivers, Well-informed foraging: Damage-released chemical cues of injured prey signal quality and size to predators, *Oecologia* **168** (2012) 651–658, Doi: 10.1007/s00442-011-2116-8.
24. M. Małogrosz, Well-posedness and asymptotic behavior for a multidimensional model of morphogen transport, *J. Evol. Equations* **12** (2012) 353–366.
25. J. D. Murray, *Mathematical Biology* (Springer-Verlag, 2002).
26. M. Negreanu and J. I. Tello, Asymptotic stability of a two species chemotaxis system with non-diffusive chemoattractant, *J. Differential Equations* **258** (2015) 1592–1617.
27. M. L. Rosenzweig and R. H. MacArthur, Graphical representation and stability conditions of predator–prey interaction, *Amer. Nat.* **97** (1963) 209–223.
28. N. Sapoukhina, Y. Tyutyunov and R. Arditi, The role of prey taxis in biological control, *Amer. Nat.* **162** (2003) 61–76.
29. R. Schaaf, Stationary solutions of chemotaxis systems, *Trans. Amer. Math. Soc.* **292** (1985) 531–556.

30. F. L. Schuster and M. Levandowsky, Chemosensory responses of *Acanthamoeba castellanii*: Visual analysis of random movement and responses to chemical signals, *J. Eukaryot. Microbiol.* **43** (1996) 150–158.
31. L. A. Segel, Analysis of population chemotaxis, in *Mathematical Models in Molecular and Cellular Biology*, eds. L. A. Segal (Cambridge Univ. Press, 1980).
32. N. Shigesada, K. Kawasaki and E. Teramoto, Spatial segregation of interacting species, *J. Theor. Biol.* **79** (1979) 83–99.
33. J. Smoller, *Shock Waves and Reaction–Diffusion Systems* (Springer, 1994).
34. Y. Tao, Global existence of classical solutions to a predator–prey model with nonlinear prey-taxis, *Nonlinear Anal. Real World Anal.* **11** (2010) 2056–2064.
35. Y. Tao and M. Winkler, Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production, to appear in *J. Eur. Math. Soc.*, arXiv: 1608.07622.
36. X. Wang, W. Wang and G. Zhang, Global bifurcation of solutions for a predator–prey model with prey-taxis, *Math. Methods Appl. Sci.* **38** (2015) 431–443, Doi: 10.1002/mma.3079.
37. X. Wang and Q. Xu, Spiky and transition layer steady states of chemotaxis systems via global bifurcation and Helly's compactness theorem, *J. Math. Biol.* **66** (2013) 1241–1266.
38. S. A. Williams and P.-L. Chow, Nonlinear reaction–diffusion models for interacting populations, *J. Math. Anal. Appl.* **62** (1978) 157–169.
39. T. D. Wyatt, *Pheromones and Animal Behavior* (Cambridge Univ. Press, 2003).
40. R. H. Yahner, *Wildlife Behavior and Conservation* (Springer, 2011).
41. M. Zuk and G. R. Kolluru, Exploitation of sexual signals by predators and parasitoids, *Quart. Rev. Biol.* **73** (1998) 418–438.