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A note on a periodic Parabolic-ODE chemotaxis system

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ABSTRACT

Existence of global classical solutions with a certain periodic asymptotic behavior of a class of reaction diffusion systems with chemotactic terms is demonstrated. This class contains a system consisting of a parabolic equation with a logistic-like term describing the behavior of a biological species and an ordinary differential equation modeling the concentration of a chemical substance throughout a regular function h. The logistic-like term limits the growth of the biological species and it contains a carrying capacity with a time-periodic asymptotic behavior.

1. Introduction

We study the prototype chemotaxis reaction–diffusion problem with logistic source containing a carrying capacity with a time-periodic asymptotic behavior:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 - u + f(x,t)), & x \in \Omega, \quad t > 0, \\ v_t = h(u,v), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} - u\chi \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1}$$

with positive initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, satisfying $0 \le u_0 \in L^\infty(\Omega) \cap W^{1,s}(\Omega)$, $\underline{v} \le v_0 \le \overline{v}$, for $x \in \Omega$ and positive constants μ, χ, $\underline{v}$ and $\overline{v}$. Here $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary of measure $|\Omega| = 1$. The growth of u is limited by a logistic-like term, where the carrying capacity presents a spatial and time dependence, $f(x,t)$, describing the resources of the environment.

Parabolic-ODE systems with chemotactic terms have been considered from the 90s, after the pioneering works of Anderson and Chaplain [1] and Levine and Sleeman [2] modeling tumor angiogenesis. Several

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authors have considered these PDEs models from a mathematical point of view to study finite time blow up or global boundedness of solutions and its asymptotic behavior. Inspired by the technics of semi-linear parabolic equations, the number of works considering parabolic-ODE systems of chemotaxis has increased considerably within last two decades, see for instance the review Horstmann [3] and references therein for more details.

We assume that $h = h(u,v) \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R})$, χ and f satisfy the following conditions for some $\varepsilon > 0$,

H1. $h_u > 0,\ h_v \le -\epsilon_v < 0,\ 0 \le h(0,0) < \mu\varepsilon/(2\chi),\ -h(0,v) \le ce^{\chi v},\ \ h_v + u\chi h_u \le -\epsilon_v/2$, with $\epsilon_v, c > 0$.

H2. For a given constant $\mathcal{C}_4 := \mathcal{C}_4(u_0, \|f\|_{L^\infty(\Omega)}, \mu, \chi, c\}$, defined in Lemma 2, with $\epsilon_0 > 0$, h fulfills $\limsup_{s\to\infty} h\left(\mathcal{C}_4 e^{\chi s}, s\right) \le -\epsilon_0$.

H3. ε fulfills $-1+\varepsilon < f(x,t)$ and there exists a periodic function f^* such that

$$\inf_{x\in\Omega} f(x,t) < f^*(t) < \sup_{x\in\Omega} f(x,t), \quad \text{and} \quad \lim_{t\to\infty} \|f(x,t) - f^*(t)\|_{L^\infty(\Omega)} \to 0, \quad \text{as} \quad t\to\infty \tag{2}$$

We denote by u^* and v^* the periodic solutions of the ODEs equations

$$u^*_t = \mu u^*(1 - u^* + f^*), \qquad \text{and} \qquad v^*_t = h(u^*, v^*) \tag{3}$$

(with f^* as in (2), see [4] for the properties of the solution u^* to the Bernoulli's equation). The above conditions cover the examples $h(r,s) = re^{-\chi s} - as$, with $a > 0$ and $h(r,s) = (re^{-\chi s} + s)/(1+s) - s$.

Theorem 1. *Let the conditions H1. to H3. hold. Then, for any nonnegative initial data* $(u_0, v_0) \in (L^\infty(\Omega) \cap W^{1,s}(\Omega))^2$, *there exists a unique global nonnegative solution* (u,v) *to system (1) uniformly bounded such that* $(u,v) \in C([0,\infty), (W^{1,s}(\Omega))^2) \cap C^1((0,\infty), (W^{1,s}(\Omega))' \times W^{1,s}(\Omega))$ *and* $\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le C < \infty$. *Moreover,* $\|u(x,t) - u^*(t)\|_{L^2(\Omega)} \to 0$ *and* $\|v(x,t) - v^*(t)\|_{L^2(\Omega)} \to 0$, *as* $t \to \infty$.

2. Proof of Theorem 1

The global existence of (u,v) over $\Omega \times (0,\infty)$ is a direct consequence of the local existence (following Theorem 6.4 in [5] and the Maximum Principle) and the uniform boundedness of (u,v) in L^∞ established in the following lemma:

Lemma 2. *Let hypotheses H1.–H3. hold. Then*

(1) *the total mass of the component* u *of the solution to (1) is bounded, i.e.,* $\int_\Omega u dx \le \mathcal{C}_1$, $0 < \mathcal{C}_1 < \infty$;

(2) *there exist positive constants* $\mathcal{C}_{2,3}$, *depending only on* $\|f\|_{L^\infty(\Omega)}$ *and* μ, *such that for all* $t \in (0, T_{max} - t_0)$ ($T_{max} > 0$ *is the maximal time*), $t_0 = \min\{1, T_{max}/2\}$, $\int_t^{t+t_0}\int_\Omega u^2 dx ds \le \mathcal{C}_2$, $\int_0^t e^{s-t}\int_\Omega u^2 dx ds \le \mathcal{C}_3$.

(3) *there exists positive constant* $\mathcal{C}_4$ *depending on* $\|u_0\|_{L^\infty}$, $\|f\|_{L^\infty}$, μ, χ *and* $\overline{v} < \infty$ *such that*

$$\left(\int_\Omega u^p \left(e^{\chi v}\right)^{1-p} dx\right)^{\frac{1}{p}} \le \mathcal{C}_4 \qquad \text{and} \quad v(x,t) < \overline{v}. \tag{4}$$

(4) *function* u *is uniformly bounded in* $L^\infty(\Omega)$ *and* $\|u\|_{L^\infty(\Omega)} \le e^{\chi\overline{v}}\mathcal{C}_4$.

Proof. The proofs of **(1)** and **(2)** are similar to that performed in [6] and [4], so we omit it here. We follow an iterative argument of Alikakos–Moser type in order to derive a bound for u. As in [6] and [7], we multiply the first equation in (1) by $u^{p-1}e^{-(1-p)\chi v}$ and integrate by parts over Ω. Since

$$\int_\Omega u^{p-1}e^{\chi(1-p)v}(\Delta u - \chi\nabla(u\nabla v))dx = -(p-1)\int_\Omega u^{p-2}e^{(1-p)\chi v}(\nabla u - \chi u\nabla v)^2 dx < 0,$$

using the properties of h, by the Mean Value Theorem and Young's inequality, for every $p \geq 1$ and a positive $\epsilon := \mu / (2\mu + 2\mu \|f\|_{L^\infty} + 2)$, we have

$$\frac{d}{dt}\int_\Omega u^p \left(e^{\chi v}\right)^{1-p} dx \leq -\int_\Omega u^p \left(e^{\chi v}\right)^{1-p} dx + \frac{\mu}{2\epsilon^{p+1}} - \frac{p\mu}{2}\int_\Omega u^{p+1}\left(e^{\chi v}\right)^{1-p} dx + c\chi(p-1)\int_\Omega u^p \left(e^{\chi v}\right)^{2-p} dx.$$

By solving the differential equation, integrating in time and eliminating the nonpositive summand it yields

$$\int_0^t e^{s-t}\int_\Omega u^{p+1}\left(e^{\chi v}\right)^{1-p} dxds \leq \mathcal{C}_5^p := \left[\max\left\{\frac{6\|u_0\|_{L^\infty}}{\min\{1,\mu\}}, \frac{3}{\epsilon^2}, \frac{6c\chi}{\mu}, \mathcal{C}_3, 1\right\}\right]^p,$$

$$\int_\Omega u^p \left(e^{\chi v}\right)^{1-p} dx \leq \mathcal{C}_4^p := \max\{3\|u_0\|_{L^\infty}, \frac{3\mu}{2\epsilon^2}, c\chi e \mathcal{C}_5\}^p.$$

In view of H2., we assume that for any $\overline{v} > \|v_0\|_{L^\infty(\Omega)}$ there exists $t_0 > 0$ such that $v(x,t_0) = \overline{v}$, which is the first t_0 fulfilling this condition. Since $v_0 < \overline{v}$, for $\overline{v}$ large enough, we have that $v_t(t_0) \leq h\left(\mathcal{C}_4 e^{\chi\overline{v}}, \overline{v}\right) < 0$. With this contradiction, the proof of **(3)** is complete.

As a consequence of the previous bounds and taking limits when $p \to \infty$ we conclude **(4)**. □

Lemma 3. *Assume H1.–H3. There exists a positive constant $\mathcal{C}_6$ such that the following estimate holds*

$$\int_\Omega |\nabla v|^2 dx \leq \mathcal{C}_6 < \infty. \tag{5}$$

Proof. Consider the functions

$$F_1 := \int_\Omega \frac{u}{u^*}dx - 1 + \ln u^* - \int_\Omega \ln u dx; \quad F_2 := \ln\left(\int_\Omega u dx\right) - \ln u^*. \tag{6}$$

Functionals of quite a similar form have previously been used in several works on related chemotaxis problems, e.g., in [8]. Notice that $F_1 \geq 0$ and $F_2 \geq c_0$. After direct calculations we have

$$\frac{d}{dt}\left(F_1 + F_2 + k\int_\Omega |\nabla v|^2 dx\right) \leq \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right) + \mu\|f^* - f\|_{L^1(\Omega)},$$

with $k := \epsilon_0/(2\mathcal{C}_4^2\|h_u\|_{L^\infty(\Omega)})$ and ϵ_0 as in H2., (for more details, see [6,9]). We divide by u^* in (3) and integrate over $(0,T)$, since

$$u^* \geq \varepsilon_1 > 0, \quad \int_0^\infty \|f^* - f\|_{L^1(\Omega)} dt \leq c < \infty, \quad \int_\Omega \ln u dx \leq \int_\Omega u dx \leq \mathcal{C}_1 < \infty$$

we obtain (5). □

Lemma 4. *Under assumptions H1.–H3. we have that there exists $\varepsilon_2 > 0$ independent of t, such that*

$$\int_\Omega u dx \geq \varepsilon_2, \quad \text{for any } t > 0. \tag{7}$$

Proof. We proceed as in Mizukami–Yokota [7] Lemma 4.2. and multiply the equation of u by $u^{-\beta}e^{\chi\beta v}$ for some $\beta \in (1,2)$. After integration by parts we obtain

$$\frac{d}{dt}\int_\Omega e^{\chi\beta v}u^{1-\beta}dx = \chi\beta\int_\Omega h(u,v)e^{\chi\beta v}u^{1-\beta}dx + (1-\beta)\beta\int_\Omega e^{\chi\beta v}u^{-1-\beta}|\nabla u - \chi u\nabla v|^2 dx$$
$$+ \mu(1-\beta)\textstyle\int_\Omega e^{\chi\beta v}u^{1-\beta}(1+f-u)dx.$$

Since $\beta \in (1,2)$, we have that

$$(1-\beta)\beta\int_\Omega e^{\chi\beta v}u^{-1-\beta}|\nabla u - \chi u\nabla v|^2 dx \leq 0.$$

Notice that, by the Mean Value Theorem it yields $h(u,v) = h(0,0) + h_u(\xi_1,0)u + h_v(u,\xi_2)v$, for some (ξ_1,ξ_2). Assumption H1. implies

$$\int_\Omega h(u,v)e^{\chi\beta v}u^{1-\beta}dx \le h(0,0)\int_\Omega e^{\chi\beta v}u^{1-\beta}dx + c_1\int_\Omega e^{\chi\beta v}u^{2-\beta}dx,$$

with $0 < c_1 < \infty$. Therefore we get

$$\frac{d}{dt}\int_\Omega e^{\chi\beta v}u^{1-\beta}dx \le \chi\beta h(0,0)\int_\Omega e^{\chi\beta v}u^{1-\beta}dx + \mu(1-\beta)\varepsilon\int_\Omega e^{\chi\beta v}u^{1-\beta}dx + [\mu(\beta-1)+c_1]\int_\Omega e^{\chi\beta v}u^{2-\beta}dx.$$

Then

$$\frac{d}{dt}\int_\Omega e^{\chi\beta v}u^{1-\beta}dx \le [\chi\beta h(0,0) - \mu(\beta-1)\varepsilon]\int_\Omega e^{\chi\beta v}u^{1-\beta}dx + c_2, \qquad c_2 > 0.$$

In view of assumption H1. for β close enough to 2, we get, by Maximum Principle that $\int_\Omega e^{\chi\beta v}u^{1-\beta}dx \le c$ and the non-negativity of v implies $\int_\Omega u^{1-\beta}dx \le \tilde{c}$, with positive constants c and $\tilde{c}$. We find a positive lower bound for $\int_\Omega udx$ as in (7) by applying the Hölder inequality, i.e.,

$$|\Omega|^{\frac{\beta}{\beta-1}}\left(\int_\Omega u^{1-\beta}dx\right)^{\frac{1}{1-\beta}} \le \int_\Omega udx$$

Similar results can be found in [10] and [11] for parabolic–elliptic and fully parabolic systems (see also [12] for parabolic-ODE systems). □

As in [4], in order to prove the convergence of the solution u to u^*, we proceed into two steps: first we get the convergence of the solution u to its average $\int_\Omega u$, to obtain later the convergence of the average to the periodic function u^*, the solution of the first equation in (3). We define the positive functions

$$K(t) := \int_\Omega\left(u - \int_\Omega udx\right)^2 dx \quad \text{and} \quad \tilde{K}(t) := \left(\int_\Omega udx - u^*\right)^2. \tag{8}$$

Lemma 5. *Assume H1.–H3. There exists a positive constant $\mathcal{C}_6$ such that the following estimate holds*

$$\int_0^\infty K(t)dt \le \mathcal{C}_6 < \infty. \tag{9}$$

Proof. Consider F_1 and F_2 as defined in (6). After direct calculations we have

$$\frac{d}{dt}\left(F_1 + F_2 + k\int_\Omega |\nabla v|^2\right) + K \le \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \mu\|f^* - f\|_{L^1(\Omega)},$$

with $k := \epsilon_0/(2\mathcal{C}_4^2\|h_u\|_{L^\infty(\Omega)})$ and ϵ_0 as in H2., (for more details, see [6,9]). We divide by u^* in (3) and integrate over $(0,T)$ to obtain a lower bound, $\varepsilon_1 \le u^*$; moreover by Lemma 4 , we have $\int_\Omega u \ge \varepsilon_2$. After integration over $(0,T)$ and taking limits when $T \to \infty$, we obtain (9). □

Lemma 6. *Let H1.–H3. hold. Then,*

$$\int_0^\infty\int_\Omega |\nabla v|^2 dxdt \le \mathcal{C}_7 < \infty, \quad \int_0^\infty\int_\Omega |\nabla u|^2 dxdt \le \mathcal{C}_8 < \infty, \quad \int_0^\infty \tilde{K}dt \le \mathcal{C}_9\int_0^\infty F_2^2 dt < \infty, \tag{10}$$

for some positive constants $\mathcal{C}_7$, $\mathcal{C}_8$ and $\mathcal{C}_9$.

Proof. We proceed as in previous lemma,

$$\frac{d}{dt}\left(F_1 + F_2 + \int_\Omega \frac{\lambda_0}{2}|\nabla v|^2\right) + K + \frac{p_0}{4}\int_\Omega \frac{\lambda_0}{2}|\nabla v|^2 \le \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \mu\|f^* - f\|_{L^1(\Omega)}, \tag{11}$$

with

$$p_0 := \frac{4\epsilon_v^2}{\mathcal{C}_4^2 \|h_u\|^2_{L^\infty(\Omega)}} \quad \text{and} \quad \lambda_0 = \frac{-2h_v - \chi u h_u}{u^2 h_u^2}$$

and

$$\int_0^\infty \int_\Omega |\nabla u|^2 dx dt \le \|u\|^2_{L^\infty(\Omega)} \int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} dx dt \le \mathcal{C}_8 < \infty;$$

moreover $d(F_2^2)/dt + 2\mu\xi F_2^2 \le |F_2|K(t) + |F_2|\mu\|f - f^*\|_{L^1(\Omega)} \le 2\mathcal{C}_4^2 (K(t) + \mu\|f - f^*\|_{L^1\Omega})$.

Then, after integration it results $\int_0^\infty F_2^2 dt < \infty$. By Lemma 2, we have $\tilde{K} \le \mathcal{C}_9 F_2^2 < \infty$. Therefore, after integration over $(0, \infty)$ in (11), we end the proof. □

Now we apply a weaker version of Lemma 5.1 in Friedman–Tello [13],

Lemma 7. *Let $k : \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying $k(t) \ge 0$, $k' \le C$ for any $t \ge 0$, and $\int_0^\infty k(s)ds \le C < \infty$, then, $k(t) \to 0$ as $t \to \infty$.*

From a direct calculation we have $K' \le \mathcal{C}_{10}$ and $\tilde{K}' \le \mathcal{C}_{11}$, with two positive constants $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$. Hence, K and $\tilde{K}$ verify the assumptions of Lemma 7 and then

$$\|u - u^*\|^2_{L^2(\Omega)} = \int_\Omega |u - u^*|^2 dx \le K + \tilde{K} \to 0, \qquad \text{as} \quad t \to \infty.$$

We denote $z = v - v^*$. Then, by the assumptions on h and in view of the convergence rate for u we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2 dx \le -\epsilon_v \int_\Omega z^2 dx + \|h_u z\|_{L^\infty(\Omega)} \int_\Omega (u - u^*)^2 dx.$$

By solving the differential inequality we get the convergence rate for v.

The uniform boundedness over $\Omega \times (0, \infty)$ and the convergence rates for the solution (u, v) to (1) immediately lead to Theorem 1. □

Remark 1. As a consequence of Theorem 1, and in view of uniform bounds of u and v we conclude

$$\|u(x,t) - u^*(t)\|_{L^p(\Omega)} \to 0 \quad \text{and} \quad \|v(x,t) - v^*(t)\|_{L^p(\Omega)} \to 0, \quad \text{as} \quad t \to \infty, \quad \text{for any} \quad p \in [1, \infty)$$

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