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On a fully parabolic chemotaxis system with source term and periodic asymptotic behavior

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Abstract. We study a parabolic–parabolic chemotactic PDE's system which describes the evolution of a biological population "u" and a chemical substance "v" in a two-dimensional bounded domain with regular boundary. We consider a growth term of logistic type in the equation of "u" in the form $u(1-u+f(x,t))$, for a given bounded function "f" which tends to a periodic in time function independent of x when t goes to infinity. We study the global existence of solutions and its asymptotic behavior for a range of parameters and initial data.

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1. Introduction

In this article, we study a system of two coupled parabolic PDE's modeling *chemotaxis*. Chemotaxis is the capability of some living organisms to direct their movement in response to the presence of a chemical gradient. This response can be either positive (*chemoattractant*) or negative (*chemorepellent*). The individuals of the biological species are able to recognize the chemical signal "v," to measure its concentration and to move in the direction of the chemical gradient. The chemotactic process appears as a common topic in biological studies, for instance the movement of some bacteria—such as *E. coli*—or the movement of human blood neutrophils, see, e.g., [3].

Mathematical models of chemotaxis appeared in 1970 with the so-called *Keller–Segel* model (see [15,16]) after Patlak [26]. A wide summary of mathematical results can be found in the surveys of Horstmann [10,11] or Bellomo et al. [4].

We consider the following parabolic–parabolic system which describes the evolution of a biological species "u" and a chemical substance "v" in a bounded domain $\Omega \subset \mathbb{R}^2$ with regular boundary:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \mathrm{div}(\chi u \nabla v) + \mu u(1+f(x,t)-u), & x \in \Omega, \quad t>0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t>0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \quad t>0, \end{cases} \tag{1.1}$$

where $f(x,t)$ converges to a homogeneous in space and periodic in time function $f^*(t)$. Here, the term $1+f$ is related with the environmental carrying capacity of the system, frequently taken as a constant. There are several examples in the nature of species with periodic behavior, for instance in the movement of the amebas *Dictyostelium discoideum* toward its center of aggregation, the medium velocity is periodic (see Steinbock, Hashimoto and Müller [27]), in Dunn and Zicha [6] it is observed periodicity in the chemotaxis of the human neutrophils and also it is referred in Zusman et al. [37] in the movement of the *Myxococcus xanthus*. Once the value $1 + f(x,t)$ is overcome by "u," the logistic term has a negative effect in the growth of the population "u."

It is well known that the solutions of the minimal-chemotaxis-logistic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \operatorname{div}(\chi u \nabla v) + au - bu^2, & x \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ do not present blow-up for any $a \in \mathbb{R}$, $\tau \geq 0$, $\chi > 0$ and $b > 0$ (see Osaki et al [25] for the two-dimensional case, Winkler [33] and Tello and Winkler [31] for the parabolic–elliptic case). Nakaguchi and Osaki in [18] consider a fully parabolic system where the logistic term presents a general form

$$\mu(u - u^\theta), \quad \text{for} \quad \theta > 1$$

and v satisfies the equation

$$\tau v_t - \Delta v + v = g(u),$$

for g defined by

$$g(u) := u(1+u)^\beta, \quad \beta \in (0,2].$$

They prove that for $n \geq 2$, the solution exists globally in time under the assumptions

$$\beta \leq \frac{\theta}{2}, \quad \beta < \frac{n+2}{2n}(\theta - 1).$$

For $\theta = \beta = 1$ and $n \geq 2$, the solutions may present blow-up. See also Tu and Qiu [32] and Winkler [36]. The one-dimensional problem

$$\begin{cases} u_t - \epsilon u_{xx} = -(uv_x)_x + \kappa u - \mu u^2, \\ 0 = v_{xx} - v + u \end{cases}$$

in a bounded domain has been studied in Winkler [35]. The author obtained global existence of solutions for $\mu \geq 1$, and some solutions blow up for $\mu < 1$ in the limit case $\epsilon = 0$. Later, Lankeit [17] extended the result for the n-dimensional radially symmetric case. In Kang and Stevens [14], the authors generalize the results to any convex domain in $\mathbb{R}^n$, bounded or unbounded for $\epsilon = 0$. In [14], the authors obtain finite time blow-up for $\mu < 1$.

In [39], the author replaces the logistic source $au - bu^2$ with a kinetic term $g(u)$ fulfilling $g(0) \geq 0$,

$$\lim \inf_{s \to \infty} \left\{ -g(s) \frac{\ln s}{s^2} \right\} = \mu_1 \in (0, \infty],$$

as well as

$$(\chi - \mu_1)^+ M < \frac{1}{2C_{\mathrm{GN}}^4},$$

where $c^+ = \max\{c, 0\}$, C_{GN} is the Gagliardo–Nirenberg constant and

$$M = ||u_0||_{L^1(\Omega)} + |\Omega| \inf_{\eta > 0} \frac{\sup\{g(s) + \eta s : s > 0\}}{\eta}.$$

In this setup, it is shown that this problem does not have any blow-up by ensuring all solutions are global-in-time and uniformly bounded. Clearly, g covers sources like $g(s) = as - bs^\theta$ with $b > 0$ and $\theta \geq 2$.

In Issa and Shen [13], the authors consider a parabolic–elliptic chemotaxis system with a logistic term in the form

$$u\left(a_1(x,t)-a_2(x,t)u-a_3(x,t)\int\limits_{\Omega} u\mathrm{d}x\right)$$

in $\mathbb{R}^n$. In [13], the case where the coefficients a_i (for $i=1,2,3$) are periodic in time is also considered. For this case, the authors obtain the existence of periodic solutions. Similar results are obtained in [20].

In [24], we assumed $\tau=0$, obtaining a parabolic–elliptic system of PDE's. We proved the existence and uniqueness of the solution of the corresponding system, under assumption

$$\chi<\mu.$$

Also we obtain, under some assumptions on the initial conditions and f, that the solution presents the following asymptotic behavior

$$\lim_{t\to\infty}\|u-u^*\|_{L^\infty(\Omega)}+\|v-u^*\|_{L^\infty(\Omega)}=0,$$

where u^* is the periodic in time function defined by

$$u^*(t)=\frac{u_0^* e^{\int\limits_0^t \mu(1+f^*(s))\mathrm{d}s}}{1+u_0^*\mu\int\limits_0^t e^{\int\limits_0^\tau \mu(1+f^*(s))\mathrm{d}s}\mathrm{d}\tau}, \tag{1.2}$$

where

$$u_0^*:=\frac{e^{\int\limits_0^T \mu(1+f^*(s))\mathrm{d}s}-1}{\mu\int\limits_0^T e^{\int\limits_0^\tau \mu(1+f^*(s))\mathrm{d}s}\mathrm{d}\tau}.$$

Notice that u^* is the solution of the equation

$$\frac{\mathrm{d}u^*}{\mathrm{d}t}=\mu u^*(1+f^*-u^*). \tag{1.3}$$

In this article, we study the problem for $\tau>0$ and obtain global existence of solution and its convergence to u^* by using an energy method instead of a comparison argument (see, for instance, [8,21–23]). Throughout the paper, we use the notation $\Omega_t:=\Omega\times(0,t)$, for $t\in(0,\infty]$ and we work under the following assumptions:

- The initial data (u_0,v_0) satisfy

$$(u_0,v_0)\in\left[C^{2+\beta}(\bar{\Omega})\right]^2, \tag{1.4}$$

for some $\beta>0$.

$$\frac{\partial u_0}{\partial n}=\frac{\partial v_0}{\partial n}=0,\quad x\in\partial\Omega. \tag{1.5}$$

$$u_0\geq 0,\quad v_0\geq 0. \tag{1.6}$$

$$\int\limits_\Omega \ln(u_0)\mathrm{d}x\geq -L>-\infty. \tag{1.7}$$

$$\int\limits_\Omega u_0\mathrm{d}x>0,\quad \int\limits_\Omega v_0\mathrm{d}x>0. \tag{1.8}$$

Notice that the first statement in (1.8) is a consequence of (1.6) and (1.7).

- There exists a positive constant $\epsilon_1 > 0$ such that

$$f(t,x) > -1 + \epsilon_1. \tag{1.9}$$

- Function f satisfies

$$f \in C^{\beta;\alpha}_{x,t}(\Omega_{x,t}), \tag{1.10}$$

  for some $\alpha > 0$.

$$\sup_{t>0} \|f\|_{L^\infty(\Omega)} := \|f\|_{L^\infty(\Omega_\infty)} < \infty, \tag{1.11}$$

$$\int_0^\infty \int_\Omega |\nabla f|^2 \mathrm{d}x\mathrm{d}t \le c < \infty, \tag{1.12}$$

$$\int_0^\infty \|f - f^*\|_{L^1(\Omega)} \mathrm{d}t \le c < \infty, \tag{1.13}$$

  where $f^* = f^*(t)$ is independent of x and periodic in time of period T.
- The coefficients χ and μ fulfill

$$\mu > \frac{\chi^2}{16} \max \left\{ \int_\Omega u_0 \mathrm{d}x, \frac{1}{\mu}(1 + \|f\|_{L^\infty(\Omega_\infty)}) \right\}. \tag{1.14}$$

- For simplicity and without loss of generality, we assume

$$|\Omega| = 1. \tag{1.15}$$

The article is organized as follows: In Sect. 2, we present the results of existence and uniqueness of solutions of system (1.1) under hypothesis (1.4)–(1.13) and (1.15). The proofs are based on Moser–Alikakos iteration method [1]; since the result is standard and similar proofs can be found in the literature (see, for instance, Winkler [33], Xiang [38]), we omit the details. In Sect. 3, we also assume (1.14) to prove that the solution to the PDE's system satisfies

$$\lim_{t\to\infty} \|u - u^*\|_{L^2(\Omega)} + \|v - v^*\|_{L^2(\Omega)} = 0,$$

in a two-step process. First, we obtain that the solution (u, v) goes to $(\tilde{u}, \tilde{v})$ as t goes to ∞, where $\tilde{u}, \tilde{v}$ are given by

$$\tilde{u} = \frac{1}{|\Omega|} \int_\Omega u \mathrm{d}x, \quad \tilde{v} = \frac{1}{|\Omega|} \int_\Omega v \mathrm{d}x,$$

respectively. Secondly, we prove that $(\tilde{u}, \tilde{v})$ converges to (u^*, v^*) where the known function u^* is defined in (1.2) and v^* is the solution to

$$\tau \frac{dv^*}{\mathrm{d}t} = u^* - v^*.$$

2. Existence and uniqueness of solutions

In this section, we present the results of global existence of solutions of (1.1). Our main result of this section is the following.

Theorem 2.1. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Let be $\tau > 0$ and $\chi \in \mathbb{R}$. Then, for all $\mu > 0$, for any nonnegative u_0 and v_0 fulfilling assumptions (1.4)–(1.13), then (1.1) possesses a uniquely determined global solution (u, v) for which both u and v are nonnegative and bounded in $\Omega \times (0, \infty)$.*

The proof follows a "Moser–Alikakos iteration method." The local existence, uniqueness, and extendibility of classical solutions to (1.1) are obtained applying the well-known results of Amann [2]. The proof is similar to the proof of [Lemma 3.3, [19]] or [Lemma 1.1, [33]], therefore we omit the details.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with regular boundary. Assume that the initial data (u_0, v_0) are nonnegative, satisfying (1.4)–(1.13) such that $0 < u_0 \in C^0(\bar{\Omega})$ and $0 < v_0 \in W^{1,\infty}(\bar{\Omega})$. Then, there exist $T_{\max} \in (0, \infty]$ and a unique pair of nonnegative functions (u, v),*

$$u \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$
$$v \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^{\infty}_{loc}([0, T_{\max}); W^{1,s}(\Omega)),$$

for $s > 2$

$$(u, v) \in \left[C^{2+\beta, 1+\frac{\beta}{2}}_{x,t}(\bar{\Omega} \times [0, T_{\max})) \right]^2,$$

which is the classical maximal solution of (1.1) on $\Omega \times [0, T_{\max})$. Furthermore, or $T_{\max} = \infty$ or, $T_{\max} < \infty$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \text{ if } t \nearrow T_{\max}.$$

We control the $W^{1,q}$-bounds of v in terms of L^p-norms of u in order to get higher-order regularity of u. For this purpose, we shall utilize the widely known smoothing $L^p - L^q$ properties of the Neumann heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ in Ω; see, for instance, [34]. Applying these heat Neumann semigroup estimates to the v-equation in (1.1), we have the following widely known lemma, [[12], Lemma 4.1], [[9] Lemma 3.4], for instance.

Lemma 2.2. *For $p \geq 1$, we consider*

$$\begin{cases} q \in \left[1, \dfrac{np}{n-p}\right), & \text{if} \quad p \leq n, \\ q \in [1, \infty], & \text{if} \quad p > n. \end{cases} \tag{2.1}$$

Then, there exists $C = C(p, q, v_0, \Omega) > 0$ such that the unique global-in-time classical solution (u, v) to (1.1) satisfies

$$||v(t)||_{W^{1,q}} \leq C(1 + \sup_{s \in (0,t)} ||u(s)||_{L^p}). \tag{2.2}$$

The following auxiliary statement is applied to obtain the existence of solutions and the asymptotic behavior. Since the proof is standard and similar to the proof of Lemma 2.3 in [28], we omit the details.

Lemma 2.3. *Let $T \leq \infty$ and α be positive constants, suppose that y is a nonnegative absolutely continuous function on $[0, T)$ satisfying*

$$\begin{cases} y' + \alpha y \leq g(t), & \text{for a.e. } t \in (0, T), \\ y(0) = y_0, \end{cases}$$

for g, a nonnegative function satisfying

$$\int_t^{t+t_0} g(s)ds \leq C, \quad \text{for all } t \in [0, T - t_0)$$

and $t_0 > 0$. *Then,*

$$y \leq \max\left\{y(0) + C, \frac{C}{\alpha} + 2C\right\}, \quad \textit{for all} \ \ t \in (0, T).$$

2.1. Basic a priori bounds for u and v

According to Theorem 2.1, in order to prove the global existence of (u, v) over $\Omega \times (0, \infty)$, we establish the uniform boundedness of (u, v) in $L^\infty(\Omega)$. First, we present the estimates of u in $L^p(\Omega)$.
Now, we quote basic properties concerning the total mass of the population and the boundedness assertions of the chemical.

Lemma 2.4. *Suppose that* (u, v) *is the solution to* (1.1)*, then, under assumptions* (1.4), (1.5) *and* (1.13)*, the solution* (u, v) *satisfies*

$$u, v \geq 0$$

$$\int\limits_\Omega u(x,t)dx \leq c_1 := c_1(u_0, ||f||_{L^\infty}, \mu, |\Omega|), \quad \forall t \in [0, T_{\max}), \tag{2.3}$$

$$\int\limits_\Omega v(x,t)dx \leq c_2 := c_2(||u||_{L^1}), \quad \forall t \in [0, T_{\max}), \tag{2.4}$$

$$\int\limits_t^{t+t_0} \int\limits_\Omega u^2(x,t)dx\,ds \leq \max\{1, t_0\}c_3, \quad \forall t \in [0, T_{\max} - t_0), \tag{2.5}$$

$$\int\limits_\Omega |\nabla v(x,t)|^2 dx \leq c_4, \quad \forall t \in [0, T_{\max}), \tag{2.6}$$

$$\int\limits_t^{t+t_0} \int\limits_\Omega |\nabla v(x,t)|^2 dx\,ds \leq \max\{1, t_0\}c_4, \quad \forall t \in [0, T_{\max} - t_0), \tag{2.7}$$

$$\int\limits_t^{t+t_0} \int\limits_\Omega |\Delta v(x,t)|^2 dx\,ds \leq \max\{1, t_0\}c_5, \quad \forall t \in [0, T_{\max} - t_0), \tag{2.8}$$

with $t_0 = \min\{1, \frac{1}{6}T_{\max}\}$ *and positive constants* c_i*,* $i = 1, \ldots, 5$.

Since the proof of the previous inequalities is standard, we omit the details; see, for instance, [33].
Based on the previous boundedness, there are many common methods to get L^2 boundedness of u,

$$\int\limits_\Omega u^2(x,t)dx \leq c_6, \quad \forall t \in (0, T_{\max}) \quad \forall t \in [0, T_{\max}) \tag{2.9}$$

for some uniform constant c_6 independent of t, i.e.,

$$c_6 = c_6(u_0, v_0, ||f||_{L^\infty}, \mu, |\Omega|, C_{\mathrm{GN}}),$$

see, for instance, [25,29,38], among others.

As a consequence of the previous estimate, after standard computations we get

$$||u(t)||_{L^3(\Omega)} \leq c_7, \tag{2.10}$$

and

$$\|\nabla v\|_{L^\infty(\Omega)} \leq c_8, \tag{2.11}$$

for some positive constants c_7 and c_8, independent of t.

The inequality

$$\|u(\cdot,t)\|_{L^q(\Omega)} \le c_q \tag{2.12}$$

is obtained by using an iterative method, for c_q independent of t.

Finally, the global boundedness is obtain by using a Moser–Alikakos iteration. The result is presented in the following lemma.

Lemma 2.5. *Under hypothesis* (1.4)–(1.5)*, there exists a positive constant* $C > 0$*, independent of* t*, such that the solution of* (1.1) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C, \quad \forall t > 0. \tag{2.13}$$

The proof is similar to the proof presented in [21], therefore we omit the details.

3. Asymptotic behavior

In this section, we address our study to asymptotic behavior of the solutions of problem (1.1). We obtain that such solution converges to a homogeneous in space and periodic in time function u^* defined in (1.2) which satisfies equation (1.3)

$$\frac{du^*}{\mathrm{d}t} = \mu u^*(1 - u^* + f^*).$$

For simplicity, we assume $|\Omega| = 1$. The result is enclosed in the following theorem.

Theorem 3.1. *Suppose* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with smooth boundary,* $\tau > 0$ *and* $\chi \in \mathbb{R}$*. Then, for all* $\mu > 0$*, for any nonnegative* u_0 *and* v_0 *fulfilling assumptions (1.4)–(1.13), the solution* (u, v) *to problem (1.1) fulfills*

$$\|u - u^*\|_{L^2(\Omega)} + \|v - v^*\|_{L^2(\Omega)} \to 0, \quad \text{as } t \to \infty, \tag{3.1}$$

where v^* *is defined by the solution to*

$$\tau\frac{dv^*}{dt} = u^* - v^*, \quad v(0) = \int_\Omega v_0(x)dx.$$

As stated in the introduction, we divide this into two steps. We start with the convergence of (u, v) to $(\tilde{u}, \tilde{v})$.

Lemma 3.1. *For* $u \in L^\infty(\Omega_\infty)$ *and* $|\nabla v| \in L^\infty(0, T_{\max} : L^2(\Omega))$*, there exists a positive constant* $c_{\frac{2}{3}} > 0$ *such that*

$$\int_\Omega u dx \ge c_{\frac{2}{3}}.$$

The proof follows the proof of Theorem 1.1. in Tao and Winkler [30], therefore we omit the details.

Lemma 3.2. *Let* u^* *be the periodic solution to* (1.3) *of period* T*, then, there exists* $\epsilon_2 > 0$ *such that*

$$u^* > \epsilon_2.$$

Proof. We divide by u^* in (1.3) and integrate over $(0, T)$ to obtain the equation

$$\frac{1}{\mu}\frac{u_t^*}{u^*} + u^* = 1 + f^* \ge \epsilon_1.$$

We define $\epsilon_2 := \frac{1}{2}\max\{u_0^*, \epsilon_1\}$ and proceed by contradiction. Suppose that there exists t_0 such that $u^*(t_0) = \epsilon_2$ and $u^*(t) > \epsilon_2$ for all $t \in (0, t_0)$ and

$$u_t^*(t_0) \leq 0. \tag{3.2}$$

We replace in the equation to obtain

$$\frac{1}{\mu}\frac{u_t^*(t_0)}{\epsilon_2} + \epsilon_2 \geq \epsilon_1,$$

equivalent to

$$\frac{1}{\mu}\frac{u_t^*(t_0)}{\epsilon_2} \geq \epsilon_1 - \epsilon_2 \geq 0,$$

which contradicts (3.2) and the proof ends. □

We now define the following function

$$k_1(t) := \int_\Omega \left(u - \int_\Omega u\mathrm{d}x\right)^2 \mathrm{d}x \tag{3.3}$$

which is clearly a nonnegative function.

Recall that for simplicity and without loss of generality, we have assumed $|\Omega| = 1$ in (1.15).

Lemma 3.3. *Under assumptions of Theorem 3.1, we have*

$$\int_0^\infty k_1(t)\,dt \leq c < \infty, \tag{3.4}$$

with a positive constant c.

Proof. We integrate the first equation of (1.1) over Ω to obtain

$$\begin{aligned}
\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x &= \int_\Omega u(1+f-u)\mathrm{d}x \\
&= \int_\Omega \left(u - \int_\Omega u\mathrm{d}x\right)(1+f-u)\mathrm{d}x + \int_\Omega u\mathrm{d}x\left(1 + \int_\Omega f\mathrm{d}x - \int_\Omega u\mathrm{d}x\right) \\
&= \int_\Omega \left(u - \int_\Omega u\mathrm{d}x\right)\left(\int_\Omega u\mathrm{d}x - u\right)\mathrm{d}x + \int_\Omega \left(u - \int_\Omega u\mathrm{d}x\right)(f - f^*)\mathrm{d}x \\
&\quad + \int_\Omega u\left(1 + f^* - \int_\Omega u\mathrm{d}x\right)\mathrm{d}x + \int_\Omega u\left(\int_\Omega (f - f^*)\mathrm{d}x\right)\mathrm{d}x.
\end{aligned}$$

Since f and f^* are uniformly bounded, we have

$$\int_\Omega \left(u - \int_\Omega u\mathrm{d}x\right)(f - f^*)\mathrm{d}x \leq \delta k_1 + c(\delta)\|f - f^*\|_{L^1(\Omega)},$$

for $\delta > 0$ satisfying

$$(1-\delta)\mu > \frac{\chi^2}{16}\max\left\{\int_\Omega u_0\mathrm{d}x, \frac{1}{\mu}(1 + \|f\|_{L^\infty})\right\}.$$

Therefore,

$$\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\mathrm{d}x \le -(1-\delta)k_1 + \int_\Omega u\left(1+f^* - \int_\Omega u\mathrm{d}x\right)\mathrm{d}x + c(\delta)\|f-f^*\|_{L^1(\Omega)}.$$

By dividing the last inequality by $\int_\Omega u\mathrm{d}x$, we arrive to

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\int_\Omega u\mathrm{d}x\right) \le \mu\left[-\frac{(1-\delta)k_1}{\int_\Omega u\mathrm{d}x} + 1 + f^* - \int_\Omega u\mathrm{d}x\right] + \frac{\mu c(\delta)}{\int_\Omega u\mathrm{d}x}\|f-f^*\|_{L^1(\Omega)}. \tag{3.5}$$

In the same fashion, we divide equation (1.3) by u^* to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln u^*) = \mu(1+f^*-u^*). \tag{3.6}$$

Now, we subtract (3.6)–(3.5)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\left(\int_\Omega u\mathrm{d}x\right) - \ln u^*\right) \le \mu\left[-\frac{(1-\delta)k_1}{\int_\Omega u\mathrm{d}x} + u^* - \int_\Omega u\mathrm{d}x\right] + \frac{\mu c(\delta)}{\int_\Omega u\mathrm{d}x}\|f-f^*\|_{L^1(\Omega)}. \tag{3.7}$$

For the sake of simplicity, let us consider the following functions

$$F_1 := \int_\Omega \frac{u}{u^*}\mathrm{d}x - 1 + \ln u^* - \int_\Omega \ln u\mathrm{d}x; \quad F_2 := \ln\left(\int_\Omega u\mathrm{d}x\right) - \ln u^*. \tag{3.8}$$

Functionals of quite a similar form have previously been used in several works on related chemotaxis problems, e.g., in [5].

Notice that $h:\mathbb{R}_+ \to \mathbb{R}\cup\{\infty\}$ defined by

$$h(s) := s - 1 - \ln s$$

satisfies $h(s) \ge 0$ for any $s>0$, and $\lim_{s\to 0^+} h(s) = +\infty$. Since

$$F_1 = \int_\Omega h\left(\frac{u}{u^*}\right)\mathrm{d}x$$

we have that $F_1 \ge 0$ and thanks to Lemma 3.1,

$$F_2 \ge -c > -\infty, \qquad \text{for any}\, t>0.$$

Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) + \mu(1+f^*-u^*) - \int_\Omega \frac{u_t}{u}\mathrm{d}x\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) + \mu(1+f^*-u^*)\\
&\quad + \int_\Omega \left[-\frac{|\nabla u|^2}{u^2} + \chi\frac{\nabla u\nabla v}{u} - \mu(1+f-u)\right]\mathrm{d}x\\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) + \mu\|f^*-f\|_{L^1(\Omega)} + \mu\left(\int_\Omega u\mathrm{d}x - u^*\right) + \frac{\chi^2}{4}\int_\Omega |\nabla v|^2\\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) + \mu\|f^*-f\|_{L^1(\Omega)} + \mu\left(\int_\Omega u\mathrm{d}x - u^*\right)\\
&\quad + \frac{\chi^2}{4}\left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega (v-\int_\Omega v\mathrm{d}x)^2\mathrm{d}x}{2}\right) + \frac{\int_\Omega (u-\int_\Omega u\mathrm{d}x)^2\mathrm{d}x}{4}\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(F_1+F_2) \le &- \frac{\mu(1-\delta)k_1}{\int_\Omega u\mathrm{d}x} + c\|f-f^*\|_{L^1(\Omega)}\\
&+ \frac{\chi^2}{16}k_1 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) - \frac{\chi^2}{8}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega (v-\int_\Omega v\mathrm{d}x)^2\mathrm{d}x}{2}\right).\end{aligned}\tag{3.9}$$

Now, since $F_1 \ge 0$, and $F_2 \ge -c$, after integration over $(0,\tau)$ we obtain

$$\left[\frac{\mu(1-\delta)}{sup_{t\in(0,\tau)}\left\{\int_\Omega u\mathrm{d}x\right\}} - \frac{\chi^2}{16}\right]\int_0^\tau k_1\mathrm{d}t \le \left[\frac{\int_\Omega u\mathrm{d}x}{u^*}\right]_0^\tau + c\mu\int_0^\tau \|f-f^*\|_{L^1(\Omega)}\mathrm{d}t + c_0 \le c.$$

Thanks to assumptions (1.14), selection of δ and Lemma 2.4, taking limits as $\tau \to \infty$, we obtain

$$\int_0^\infty k_1\mathrm{d}t \le c < \infty \tag{3.10}$$

and the proof ends. □

Lemma 3.4. *Under assumptions of Theorem 3.1, there exists a positive constant $c < \infty$ such that*

$$\int_0^\infty\int_\Omega |\Delta v|^2 + |\nabla v|^2\,dx\,dt \le c.$$

Proof. First we notice that, after integration in the second equation in (1.1), it yields

$$\int_\Omega u\mathrm{d}x = \int_\Omega v\mathrm{d}x + \tau\int_\Omega v_t\mathrm{d}x.$$

Then, we have that

$$\tau v_t - \Delta v + v - \int_\Omega v\mathrm{d}x = u - \int_\Omega u\mathrm{d}x + \tau \int_\Omega v_t\mathrm{d}x.$$

We multiply by $-\Delta v$ and integrate over Ω the previous equation, taking into account the following identities

$$-\int_\Omega \left(v_t - \int_\Omega v_t\right)\Delta v\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\nabla v|^2\mathrm{d}x;$$

$$-\int_\Omega \left[\Delta v\left(v - \int_\Omega v\mathrm{d}x\right)\right]\mathrm{d}x = \int_\Omega |\nabla v|^2\mathrm{d}x$$

and

$$-\int_\Omega \left[\Delta v\left(u - \int_\Omega u\mathrm{d}x\right)\right]\mathrm{d}x \le \frac{1}{2}\int_\Omega |\Delta v|^2\mathrm{d}x + \frac{1}{2}\int_\Omega \left|u - \int_\Omega u\mathrm{d}x\right|^2\mathrm{d}x$$

we get the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\tau}{2}\int_\Omega |\nabla v|^2\mathrm{d}x + \frac{1}{2}\int_\Omega |\Delta v|^2\mathrm{d}x + \int_\Omega |\nabla v|^2\mathrm{d}x \le \int_\Omega \left|u - \int_\Omega u\mathrm{d}x\right|^2\mathrm{d}x.$$

After integration over $(0, \infty)$ and thanks to Lemma 3.3, we obtain the wished result. ☐

Lemma 3.5. *Under assumptions of Theorem 3.1, there exists a positive constant $c < \infty$, such that the following inequality holds*

$$\int_0^\infty\int_\Omega |\nabla u|^2\,dx\,dt \le c.$$

Proof. As in Lemma 3.3, we derivate $F_1 + F_2$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(F_1+F_2) &\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) + \int_\Omega \left[-\frac{|\nabla u|^2}{u^2} + \chi\frac{\nabla u\nabla v}{u}\right]\mathrm{d}x\\ &\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_\Omega u\mathrm{d}x}{u^*}\right) - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + \frac{\chi^2}{2}\int_\Omega |\nabla v|^2\mathrm{d}x.\end{aligned}$$

After integration, in view of Lemma 3.4, we obtain

$$\int_0^\infty\int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x\mathrm{d}t \le c < \infty.$$

In view of the boundedness of u, we have

$$\int_0^\infty\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}t \le \|u\|^2_{L^\infty(\Omega)}\int_0^\infty\int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x\mathrm{d}t \le c < \infty$$

and the proof ends. ☐

Lemma 3.6. *We assume that the hypotheses of Theorem 3.1 are fulfilled. There exists a positive constant* $c < \infty$*, independent of* t *such that the following holds*

$$\int_\Omega |\nabla u|^2 dx \le c.$$

Proof. We multiply the first equation (1.1) by $-\Delta u$ and integrate by parts to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |\Delta u|^2 \mathrm{d}x = \chi \int_\Omega \Delta u[\nabla u \nabla v + u\Delta v]\mathrm{d}x - \mu \int_\Omega \Delta u u(1+f-u)\mathrm{d}x.$$

Since ∇v is uniformly bounded by (2.11), we have

$$\chi \int_\Omega \Delta u[\nabla u \nabla v + u\Delta v]\mathrm{d}x \le \frac{1}{2}\int_\Omega |\Delta u|^2 \mathrm{d}x + c_9 \int_\Omega |\nabla u|^2 \mathrm{d}x + c_9 \int_\Omega |\Delta v|^2 \mathrm{d}x$$

and

$$-\mu \int_\Omega u(1+f-u)\Delta u \mathrm{d}x \le \frac{c_{10}}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + c_{10}\int_\Omega \nabla u \nabla f \mathrm{d}x \le c_{10}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{c_{10}}{2}\int_\Omega |\nabla f|^2 \mathrm{d}x,$$

with c_9 and c_{10} positive constants independent of t. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\Delta u|^2 \mathrm{d}x \le (c_9 + c_{10})\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{c_{10}}{2}\int_\Omega |\nabla f|^2 \mathrm{d}x + c_9 \int_\Omega |\Delta v|^2 \mathrm{d}x$$

After integration and thanks to assumption (1.12) and previous lemmas, we get the result. □

Lemma 3.7. *Let* k_1 *be defined in* (3.3)*, then, under assumptions* (1.4)–(1.14)*, there exists a positive constant* $\tilde{c}_2 < \infty$ *such that*

$$|k_1'| \le \tilde{c}_2, \qquad \textit{for } t > 0.$$

Proof. In view of Lemma 2.1, we have that $k_1 \in C^1(0,\infty)$. Then, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \left(u - \int_\Omega u \mathrm{d}x\right)^2 \mathrm{d}x = \int_\Omega u_t \left(u - \int_\Omega u \mathrm{d}x\right)\mathrm{d}x,$$

$$\int_\Omega u_t \left(u - \int_\Omega u \mathrm{d}x\right)\mathrm{d}x = -\int_\Omega |\nabla u|^2 \mathrm{d}x + \chi \int_\Omega u \nabla u \nabla v \mathrm{d}x + \int_\Omega \left(u - \int_\Omega u\mathrm{d}x\right) u(1+f-u)\mathrm{d}x.$$

Then, by applying the Young's inequality we have

$$\left| -\int_\Omega |\nabla u|^2 \mathrm{d}x + \chi \int_\Omega u\nabla u \nabla v \mathrm{d}x \right| \le \int_\Omega |\nabla u|^2 \mathrm{d}x + c\|u\|^2_{L^\infty(\Omega_\infty)}\int_\Omega |\nabla v|^2 \mathrm{d}x.$$

Boundedness of u, assumption (1.11), Lemmas 2.1 and 3.6 imply the result. □

As a consequence of Lemma 3.3 and 3.7, we obtain the asymptotic behavior of the solutions by applying Lemma 5.1 in Friedman–Tello [7]. For readers' convenience, we reproduce the statement.

Lemma 3.8. (Lemma 5.1 Friedman–Tello [7]). *Let* $k : [0,\infty) \to R$ *be a* C^1 *function such that* *(i)* $k(t) \geq 0$ *and* $k(t) \leq C_0 < \infty$ *for some constant* $C_0 > 0$ *in* $[0, \infty)$*;*

(ii) $\int\limits_0^\infty k(t)dt \leq C_1 < \infty$*;*

(iii) $|k'| \leq C_2 < \infty$ *for some constant* $C_2 > 0$ *in* $[0, \infty)$*.*

Then $k(t) \to 0$*, as* $t \to \infty$*.*

Proof of Theorem 3.1. We consider k_1 defined in (3.3), then, thanks to Lemma 3.7 we have that the function $k_1 \in C^{1+\alpha}$, for some $\alpha \in (0, 1)$. Due to Lemmas 3.3 and 3.7, we obtain that

$$\|u - \int\limits_\Omega u\mathrm{d}x\|_{L^2(\Omega)} \to 0, \qquad \text{as} \qquad t \to \infty. \tag{3.11}$$

Now, we define k_2 as follows

$$k_2(t) := \left(\int\limits_\Omega u\mathrm{d}x - u^* \right)^2$$

and consider $F_2(t)$ defined in (3.8) by

$$F_2(t) = \ln \int\limits_\Omega u\mathrm{d}x - \ln u^*.$$

So, by derivation we get

$$\frac{\mathrm{d}}{\mathrm{d}t} F_2 = \frac{\int\limits_\Omega u_t \mathrm{d}x}{\int\limits_\Omega u\mathrm{d}x} - \frac{u_t^*}{u^*},$$

which implies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F_2 &= \mu \left(\frac{\int\limits_\Omega (uf)\mathrm{d}x}{\int\limits_\Omega \mathrm{d}x} - \frac{\int\limits_\Omega u^2 \mathrm{d}x}{\int\limits_\Omega u\mathrm{d}x} + u^* - f^* \right), \\ &= \mu \left(\frac{\int\limits_\Omega u(f - f^*)\mathrm{d}x}{\int\limits_\Omega u\mathrm{d}x} - \frac{\int\limits_\Omega u(u - u^*)\mathrm{d}x}{\int\limits_\Omega u\mathrm{d}x} \right) \end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} F_2 + \frac{\mu u^*}{\int\limits_\Omega u\mathrm{d}x} \left(\int\limits_\Omega u\mathrm{d}x - u^* \right) = \mu \frac{\int\limits_\Omega u(f - f^*)\mathrm{d}x}{\int\limits_\Omega u\mathrm{d}x} - \frac{\mu k_1}{\int\limits_\Omega u\mathrm{d}x}.$$

We multiply by F_2 the previous inequality to obtain,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} F_2^2 + \frac{\mu u^*}{\int\limits_\Omega u\mathrm{d}x} \left(\int\limits_\Omega u\mathrm{d}x - u^* \right) F_2 \leq |F_2| k_1(t) + c\|f - f^*\|_{L^1(\Omega)}. \tag{3.12}$$

Thanks to mean value theorem, we claim

$$\left(\int\limits_\Omega u\mathrm{d}x - u^* \right) F_2 = \xi F_2^2,$$

for some $\xi \in \left[u^*, \int\limits_\Omega u\mathrm{d}x\right]$ if $u^* < \int\limits_\Omega u\mathrm{d}x$, or $\xi \in \left[\int\limits_\Omega u\mathrm{d}x, u^*\right]$, otherwise. As a consequence of Lemma 3.1 and Lemma 3.2, we get

$$\frac{\mu u^*}{\int\limits_\Omega u\mathrm{d}x}\left(\int\limits_\Omega u\mathrm{d}x - u^*\right) F_2 \geq cF_2^2$$

for some positive constant c. After integration in (3.12), it results

$$\int\limits_0^\infty F_2^2\mathrm{d}t \leq c < \infty.$$

Notice that, due to Lemma 2.4, we have

$$k_2 \leq cF_2^2,$$

for some positive constant c and it implies with the previous bound of F_2^2 that

$$\int\limits_0^\infty k_2\mathrm{d}t \leq c < \infty. \tag{3.13}$$

In view of Lemma 2.4, (1.11) and (1.2), we have that

$$\left|\int\limits_\Omega u\mathrm{d}x\right| < c, \quad |u^*| < |u_0^*| + 1 + \|f\|_{L^\infty(\Omega_\infty)},$$

$$|u_t^*| \leq \mu(|u_0^*| + 1 + \|f\|_{L^\infty(\Omega_\infty)})(1 + \|f\|_{L^\infty(\Omega_\infty)})$$

and

$$\left|\int\limits_\Omega u_t\mathrm{d}x\right| \leq \mu\|u\|_{L^\infty(\Omega_\infty)}((1 + \|f\|_{L^\infty(\Omega_\infty)} + \|u\|_{L^\infty(\Omega_\infty)}).$$

Since

$$k_2' = 2\left(\int\limits_\Omega u\mathrm{d}x - u^*\right)\left(\int\limits_\Omega u_t\mathrm{d}x - u_t^*\right),$$

it is easy to see that

$$|k_2'| \leq c < \infty. \tag{3.14}$$

Now, by [Lemma 5.1 Friedman–Tello [7]], (3.13) and (3.14) we obtain

$$k_2 \to 0 \quad \text{as} \quad t \to \infty. \tag{3.15}$$

Since

$$\int\limits_\Omega |u - u^*|^2\mathrm{d}x \leq k_1 + k_2,$$

by relations (3.11) and (3.15), we get

$$\|u - u^*\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty.$$

To obtain

$$\|v - v^*\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty,$$

we proceed as before and we define

$$k_3 := \int_\Omega |v-v^*|^2 \mathrm{d}x.$$

We take squares in both sides of the equation

$$\tau \frac{\mathrm{d}}{\mathrm{d}t}(v-v^*) - \Delta v + (v-v^*) = u-u^*$$

and integrate over Ω

$$\int_\Omega \left|\tau \frac{\mathrm{d}}{\mathrm{d}t}(v-v^*)\right|^2 \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\tau \left[\frac{1}{2}\int_\Omega |v-v^*|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x\right] + \int_\Omega |\Delta v|^2 \,\mathrm{d}x + \int_\Omega |\nabla (v-v^*)|^2 \,\mathrm{d}x + \int_\Omega |(v-v^*)|^2 \,\mathrm{d}x \le \int_\Omega |u-u^*|^2 \mathrm{d}x.$$

After integration in time, we claim

$$\int_0^\infty \int_\Omega \left|\tau \frac{\mathrm{d}}{\mathrm{d}t}(v-v^*)\right|^2 \mathrm{d}x\mathrm{d}t + \frac{\tau}{2}k_3 + \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_0^\infty k_3 \mathrm{d}t \le c,$$

i.e.,

$$\int_0^\infty k_3 \mathrm{d}t \le c. \tag{3.16}$$

We multiply by $-\Delta v_t$ and integrate over $\Omega \times (0,\infty)$ the second equation of (1.1) to obtain

$$\frac{\tau}{2}\int_0^\infty \int_\Omega |\nabla v_t| \mathrm{d}x\mathrm{d}t + \int_\Omega |\Delta v|^2 \mathrm{d}x + \int_\Omega |\nabla v|^2 \mathrm{d}x \le 2\int_0^\infty \int_\Omega |\nabla u|^2 \mathrm{d}x\mathrm{d}t + c(u_0) < c.$$

In the same fashion, it yields

$$\int_\Omega \left|\tau \frac{\mathrm{d}}{\mathrm{d}t}(v-v^*)\right|^2 \mathrm{d}x \le c\int_\Omega |\Delta v|^2 \,\mathrm{d}x + c\int_\Omega |v-v^*|^2 \,\mathrm{d}x + c\int_\Omega |u-u^*|^2 \mathrm{d}x \le c$$

and therefore

$$|k_3'| \le \frac{1}{2}\int_\Omega \left|\tau \frac{\mathrm{d}}{\mathrm{d}t}(v-v^*)\right|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |(v-v^*)|^2 \,\mathrm{d}x \le c. \tag{3.17}$$

Relations (3.16), (3.17) and Lemma 5.1 in Friedman–Tello [7] end the proof. □

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