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On a Parabolic-Elliptic chemotaxis system with periodic asymptotic behavior

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We study a parabolic-elliptic chemotactic PDEs system which describes the evolution of a biological population "u" and a chemical substance "v" in a bounded domain $\Omega \subset \mathbb{R}^n$. We consider a growth term of logistic type in the equation of "u" in the form $\mu u(1 - u + f(t, x))$. The function "f", describing the resources of the systems, presents a periodic asymptotic behavior in the sense

$$\lim_{t\to\infty} \sup_{x\in\Omega} |f(x,t) - f^*(t)| = 0$$

where f^* is independent of x and periodic in time. We study the global existence of solutions and its asymptotic behavior. Under suitable assumptions on the initial data and f^*, if the constant chemotactic sensitivity χ satisfies

$$\chi < \mu/2$$

we obtain that the solution of the system converges to a homogeneous in space and periodic in time function.

Keywords: Parabolic-Elliptic systems of PDEs, Chemotaxis, asymptotic behavior, periodic solutions, sub- and super-solutions, rectangle method

1. Introduction

Chemotaxis is the process under which some living organisms (such as bacterias, cells of the immune system, cells of the endothelium etc.) direct its movement in the direction of a chemical gradient. The individuals of the biological species are able to recognize the chemical signal "v", to measure its concentration and to move towards the higher concentrations of the substance (positive taxis) or away from it (negative taxis). The phenomena was discovered at the end of the XIX century with the invention of the microscope and since then, it has been widely studied.

Since 1970s, chemotaxis has been studied from a mathematical point of view, starting with the first models of PDEs suggested by Keller and Segel, [8], [9]. The mathematical bibliography is extensive, more details about obtained results in the area during the last decades can be found in D.Horstmann [6] and [7]. From the point of view of applications, mathematical models with chemotactic terms have been applied to model *Angiogenesis*, a key process in Tumor Growth, whereby endothelial cells move towards the tumor following a chemical gradient, creating new blood vessels and providing extra supply to the tumor. The process has been largely studied from the 70′s and mathematical models of PDEs have been used to describe the creation of new blood vessels (see for instance Anderson and Chaplain [1], Levine, Sleeman and Nilsen-Hamilton [10], Holmes and Sleeman [5]). The mathematical model we present in this article includes a space dependence of the chemical secretion denoted by f, although it is not the purpose of this article to study the controllability of angiogenesis systems, the term f can be considered as a control term used as a therapy, where the application presents a periodic behaviour, see for instance Delgado et al [3] where the authors have considered a flux therapy application of an anti-angiogenic therapy. Chemotaxis terms also appear in *Astrophysics* to describe gravitational interaction of particles on the gravitational equilibrium of polytropic stars (see Biler [2]), in *ecology*, to describe the attraction of predators to certain chemical signals (pheromons) of the prey (see Tello, Wrzosek [21]), in *morphogenesis*, the creation of shapes and organs in embryonic development, see Stinner, Tello and Winkler [19] among others.

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The model studied in this paper is a parabolic - elliptic system which describes the evolution of a biological species "u" and a chemical substance "v" in a bounded domain $\Omega \subset \mathbb{R}^n$. The original system in [9] is expressed in the following way:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + a(u,x,t)u, & x \in \Omega, \quad t > 0 \\ \epsilon v_t - \Delta v = g(u,v), & x \in \Omega, \quad t > 0 \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & x \in \partial\Omega, \quad t > 0 \end{cases}$$

where "a" describes the growth rate of "u". It is natural that in unlimited resources environments "a" is assumed constant, nevertheless, it is expected that the resources decreases as the population growths and the mortality increases if the population is higher than a threshold value in limited resources scenarios. One of the widely used terms to describe the population growth is a logistic type function i.e.

$$a(u,x,t)u = \mu u(N(x,t) - u), \tag{1.1}$$

where N describes the carrying capacity of the system (see for instance Murray [11] for more details). Under the threshold value N, the logistic term induces to the variable "u" to increase and over N to decrease. It is natural that N depends on time and presents some kind of periodicity caused, for instance, by the seasons of the year which may cause a seasonal behavior in the population. Several examples of this kind of periodicity are commom in the literature, in the movement of the amebas *Dictyostelium discoideum* towards its centre of aggregation, the medium velocity is periodic, see for instance Steinbock, Hashimoto and Müller [22]. In Dunn and Zicha [23], it is also observed the periodicity in the chemotaxis of the human neutrophils. For the sake of simplicity we denote

$$N(x,t) = 1 + f(x,t).$$

We assume now that diffusion of the biologic species is negligible as compared with diffusion of the chemical substance. Hence, in the time scale of the problem, ϵ is very small. We simplify the problem and take $\epsilon = 0$ to obtain

$$-\Delta v = g(u,v), \quad x \in \Omega.$$

We also simplify the term g and take a lineal expression, i.e.

$$g(u,v) = g_0 u - g_1 v,$$

where g_0 represents the ratio of production of the chemical substance and g_1 is the degradation rate of the chemical. Without lose of generality we assume $g_0 = g_1 = 1$. Hence, the system becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ -\Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Throughout the paper we work under the following hypothesis:

- The initial data u_0 satisfies

$$u_0 \in C^{0,\alpha}(\bar{\Omega}), \qquad \text{for some } \alpha \in (0,1), \tag{1.3}$$

  and

$$\frac{\partial u_0}{\partial n} = 0, \quad x \in \partial\Omega. \tag{1.4}$$

- There exist positive constants $\overline{u}_0$ and $\underline{u}_0$ such that

$$0 < \underline{u}_0 \le u_0(x) \le \overline{u}_0 < \infty. \tag{1.5}$$

- Function $f \in C(0,\infty : L^\infty(\Omega))$ satisfies

$$\int_0^\infty |\sup_{x\in\Omega}\{f(x,t)\} - \inf_{x\in\Omega}\{f(x,t)\}| dt \le C < \infty; \tag{1.6}$$

  there exists $\epsilon_1 > 0$ such that

$$f(x,t) > -1 + \epsilon_1, \tag{1.7}$$

$$|\sup_{x\in\Omega}\{f(x,t)\} - \inf_{x\in\Omega}\{f(x,t)\}| \to 0, \qquad \text{as} \quad t \to \infty. \tag{1.8}$$

  and

$$\sup_{t>0} \|f(x,t)\|_{L^\infty(\Omega)} := \|f(x,t)\|_{L^\infty(\Omega_\infty)} < \infty. \tag{1.9}$$

- There exists a periodic function f^* of period $T > 0$,

$$f^* : \mathbb{R}_+ \to \mathbb{R} \tag{1.10}$$

such that

$$\inf_{x\in\Omega}\{f(x,\cdot)\} \le f^*(\cdot) \le \sup_{x\in\Omega}\{f(x,\cdot)\}. \tag{1.11}$$

Notice that, as a consequence of (1.8) and (1.11), f^* is unique.

The paper is organized as follows: in Section 2 we study the existence and uniqueness of the solutions to (1.2) under assumption

$$\chi < \mu. \tag{1.12}$$

In Section 3 we introduce a system of ODEs associated to the PDEs system. The solutions of the ODEs are taken as sub and super-solutions of (1.2). The asymptotic behavior of the ODEs system is study for

$$2\chi < \mu \tag{1.13}$$

and shows that the solution to the PDEs satisfies

$$\lim_{t\to\infty} \left(\|u(\cdot,t) - u^*(t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - u^*(t)\|_{L^\infty(\Omega)}\right) = 0,$$

where $u^*(t)$ is a known periodic function (Theorem 4.2).

2. Existence and uniqueness of solutions

In this section we study the existence and uniqueness of global-in-time solutions to (1.2). The results are enclosed in the following theorem.

Theorem 2.1. *Under assumptions (1.3)-(1.5), (1.9) and (1.12), there exists a unique classical solution (u, v) to (1.2) such that*

$$u, v \in C^{2,1}_{loc}(\bar\Omega \times [0,\infty)),$$

and fulfills

$$0 \le u \le \tilde k, \qquad 0 \le v \le \tilde k$$

for

$$\tilde k = \max\left\{\|u_0\|_{L^\infty}, \mu\frac{1+\|f\|_{L^\infty}}{\mu-\chi}\right\}.$$

We first notice that, for any solution of (1.2) the total mass is uniformly bounded, i.e.,

$$\int_\Omega u(x,t)dx \le C < \infty. \tag{2.14}$$

The inequality is obtained after integration in (1.2), applying Cauchy-Schwarz inequality and Gronwall lemma.

The cross diffusion term in (1.2) is also expressed in the following way

$$\chi div\cdot(u\nabla v) = \chi\nabla u\cdot\nabla v + \chi u\Delta v = \chi\nabla u\cdot\nabla v - \chi u(u-v).$$

Hence, the equation for u becomes

$$u_t - \Delta u = -\chi\nabla u\nabla v + \chi u(u-v) + \mu u(1-u+f(x,t)), \quad x\in\Omega, \quad t>0. \tag{2.15}$$

We now proceed to prove Theorem 2.1.

Proof of Theorem 2.1.
Let $\tilde u(t)$ be the solution to the ordinary differential equation

$$\tilde u_t(t) = (\chi-\mu)\tilde u^2(t) + a_1\tilde u(t) \quad \text{in } (0,T), \qquad \tilde u(0) = \tilde u_0 := \|u_0(x)\|_{L^\infty(\Omega)}, \tag{2.16}$$

where

$$a_1 := 1 + \|f(x,t)\|_{L^\infty(\Omega_\infty)}.$$

Equation (2.16) is a Bernoulli differential equation, and its solutions is given by

$$\tilde{u}(t) = \left[\tilde{u}_0^{-1} e^{-a_1 t} + \int_0^t (\mu - \chi) e^{a_1 (s-t)} ds\right]^{-1},$$

which satisfies

$$\tilde{u}(t) \leq \max\left\{ \|u_0\|_{L^\infty}, \mu \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\mu - \chi} \right\}, \qquad \forall t > 0$$

We now construct a fixed point argument, we consider $T_1 < \infty$, and $p \in (n, \infty)$, and the set

$$G := \left\{ u \in L^p((0, T_1); C^0(\bar{\Omega})) \quad \text{such that} \quad 0 \leq u(x,t) \leq \tilde{u}(t) \text{ a.e. in } \Omega \times [0, T_1) \right\},$$

and the function $J : G \to L^p((0, T_1); C^0(\bar{\Omega}))$ defined by $J(u) := U$, where U is the solution to the differential equation

$$\begin{cases} U_t = \Delta U - \chi \nabla U \cdot \nabla V + U g(V, f) + (\chi - \mu) U u & \text{in } \Omega \times (0, T_1), \\ \dfrac{\partial U}{\partial n} = 0, \qquad \text{in } \partial\Omega \times (0, T_1) \\ U(x, 0) = u_0(x), \qquad \text{in } \Omega, \end{cases} \tag{2.17}$$

where V is the unique solution to

$$\begin{cases} -\Delta V + V = u & \text{in } \Omega \times (0, T_1), \\ \dfrac{\partial V}{\partial n} = 0, & \text{in } \partial\Omega \times (0, T_1) \end{cases} \tag{2.18}$$

and

$$g(V, f) := -\chi V + \mu(1 + f(t, x)). \tag{2.19}$$

Since $u \in G$, we have that $V \in L^p([0, T_1) ; W^{2,q}(\Omega))$ (for $q < \infty$) and thanks to the Maximum Principle, we obtain

$$0 \leq V(x,t) \leq \tilde{u}(t). \tag{2.20}$$

Furthermore, since $u \leq \tilde{u}$ and $u \geq 0$ it results

$$\nabla V \in [L^\infty(\Omega_{T_1})]^n$$

and

$$g(V, f) \in L^\infty(\bar{\Omega} \times [0, T_1)).$$

Thanks to Maximum Principle we get

$$0 \leq U \tag{2.21}$$

as far as the solution is defined.

The proof of the existence of solutions of (2.17) is obtained by using a compactness method for the equation

$$U_t = \Delta U - \chi \nabla U \cdot \nabla V + U g(V, f) + (\chi - \mu) U \tilde{u}, \quad x \in \Omega, \quad t \in (0, T_1), \tag{2.22}$$

for a given V and $\tilde{u} \in G$. We notice that (2.22) has a unique solution satisfying

$$U \in L^p((0, T_1); W^{2,q}(\Omega)) \cap W^{1,2}((0, T_1); L^q(\Omega)),$$

see for instance Quittner and Souplet [18] Example 51.4 and Remark 51.5. Maximum principle give us

$$0 \leq U \leq \tilde{u}.$$

Since $W^{2,q}(\Omega) \subset C^0(\bar{\Omega})$ is a compact embedding and $U_t \in L^2(\Omega \times (0, T_1))$, thanks to Aubin lemma, we have that G is a relatively compact subset of $L^p((0, T_1); C^0(\bar{\Omega}))$. Applying Schauder's fixed point theorem, we obtain the existence of, at least, a solution in G. This fixed point is a solution to (2.17). Uniqueness is also obtained by contradiction in view of the regularity of the semilinear term.

Since $\tilde{u} \in C^1(0, T_1)$ and it is independent of x, $\tilde{u}$ satisfies the equation

$$\tilde{u}_t = \Delta \tilde{u} - \chi \nabla \tilde{u} \nabla V + \mu \tilde{u}(1 + \|f\|_{L^\infty(\Omega_{T_1})}) + (\chi - \mu) \tilde{u}^2.$$

Therefore $U - \tilde{u}$ satisfies

$$\begin{aligned}
&(U-\tilde{u})_t - \Delta (U-\tilde{u}) + \chi \nabla (U-\tilde{u}) \cdot \nabla V \\
&= U(-\chi V + \mu(1+f(t,x))) - \mu \tilde{u}(1+\|f\|_{L^\infty(\Omega_{T_1})}) + (\chi-\mu)(U^2-\tilde{u}^2) \\
&= (U-\tilde{u})\left[-\chi V + \mu(1+f(t,x)) + (\chi-\mu)(U+\tilde{u})\right] + \tilde{u}(-\chi V + \mu(f(t,x) - \|f\|_{L^\infty(\Omega_{T_1})})).
\end{aligned}$$

Since $V \geq 0$ we deduce

$$-\tilde{u}\chi V \leq 0,$$

and

$$\mu\tilde{u}(f(t,x) - \|f\|_{L^\infty(\Omega_{T_1})}) \leq 0,$$

then, we have the inequality

$$(U-\tilde{u})_t - \Delta (U-\tilde{u}) + \chi \nabla (U-\tilde{u}) \cdot \nabla V \leq (U-\tilde{u})\left[-\chi V + \mu(1-u+f(t,x)) + \mu(U+\tilde{u})\right].$$

Applying Maximum Principle to $U - \tilde{u}$, in view of $U_0 \leq \tilde{u}_0$, it follows that

$$U \leq \tilde{u}.$$

Hence, U is uniformly bounded in $(0, T_1)$. As before, thanks to Aubin Lemma and and Schauder's fixed point theorem, we conclude that J has a fixed point in $L^p((0,T_1);C^0(\bar{\Omega}))$, which is a solution to the system (1.2).

Uniqueness of solutions is obtained by contradiction, since the proof is standard we omit the details (see for instance Tello and Winkler [20] Theorem 2.1). To end the proof of the Theorem we take limits when $T_1 \to \infty$. □

Remark 2.1. *The global existence of solutions is proven under assumption (1.12) for any dimension n, nevertheless (1.12) can be relaxed to*

$$\begin{cases} \mu \geq 0, & n = 1, \\ \mu > \frac{n-2}{n}\chi, & n \geq 2. \end{cases}$$

The proof is similar to the proof of Theorem 2.5 in Tello and Winkler [20], see also Galakhov, Salieva and Tello [4].

3. Qualitative properties

In this section we consider a system of ordinary differential equations associated to the nonlinear system of PDE's (1.2) to obtain the asymptotic behavior of its solutions. After it, we relate and compare the properties of the solutions of the initial PDE's system (1.2) and the ODE's system (3.23).

3.1. Associated ODE system

Recall that we need a relation of order to bound the solution of problem (1.2), (u, v), between the solutions of the new ODE's system. In Lemma 3.1 and Lemma 3.2 we prove that both solutions of the ODE's system have the same limit and, hence, also any function between them. We obtain an explicit expression of such limit in terms of f^*.

For the sake of simplicity, let us introduce the following notation that we use in the remainder of the article

$$\overline{f}(t) := \sup_{x\in\Omega}\{f(x,t)\}, \qquad \underline{f}(t) := \inf_{x\in\Omega}\{f(x,t)\}.$$

In order to prove the convergence and the stability of solutions of system (1.2), we introduce two auxiliary functions, $(\overline{u}, \underline{u}) = (\overline{u}(t), \underline{u}(t))$, defined as solutions of the initial value problem

$$\begin{cases} \overline{u}_t(t) = \chi\overline{u}(t)(\overline{u}(t) - \underline{u}(t)) + \mu\overline{u}(t)\left(1 - \overline{u}(t) + \overline{f}(t)\right), & t > 0, \\ \underline{u}_t(t) = \chi\underline{u}(t)(\underline{u}(t) - \overline{u}(t)) + \mu\underline{u}(t)\left(1 - \underline{u}(t) + \underline{f}(t)\right), & t > 0, \\ \overline{u}(0) = \overline{u}_0 \quad \underline{u}(0) = \underline{u}_0, \end{cases} \tag{3.23}$$

where the initial data satisfies

$$0 < \underline{u}_0 < \overline{u}_0 < \infty. \tag{3.24}$$

Now we study the properties of the solutions of the above system, i.e., we find a relationship between solutions $(\overline{u}(t), \underline{u}(t))$ when the initial data $(\overline{u}_0, \underline{u}_0)$ satisfies (3.24) and show that the initial ordering is inherited by the solution. Furthermore we prove that $(\overline{u}(t), \underline{u}(t))$ are actually global in time and bounded.

Lemma 3.1. *The solution to the system (3.23) exists in* $(0,\infty)$. *Moreover, for every positive bounded initial data* $\overline{u}_0$ *and* $\underline{u}_0$ *verifying (3.24), the solution* $(\overline{u}(t),\underline{u}(t))$ *satisfies the order relation*

$$0<\underline{u}(t)<\overline{u}(t)\le \max\{\overline{u}_0, 1+\|\overline{f}(t)\|_{L^\infty},\} \quad \text{for any} \quad t<\infty. \tag{3.25}$$

Proof. First we notice that the right-hand side terms in (3.23), i.e., $\chi\overline{u}(t)(\overline{u}(t)-\underline{u}(t))+\mu\overline{u}(t)(1-\overline{u}(t)+\overline{f}(t))$ and $\chi\underline{u}(t)(\underline{u}(t)-\overline{u}(t))+\mu\underline{u}(t)(1-\underline{u}(t)+\underline{f}(t))$, are continuous and locally Lipschitz in $\overline{u}(t)$ and $\underline{u}(t)$. Furthermore, we claim that (3.23) is locally well-posed and, therfore, there exists an unique solution for $t\in(0,T_{max})$ such that, if $T_{max}<\infty$, we have $|\overline{u}(T_{max})|+|\underline{u}(T_{max})|=\infty$.
Notice that the first equation of (3.23) can be written in the form

$$\overline{u}_t(t)=\overline{u}(t)h(\overline{u}(t),\underline{u}(t),\overline{f}(t)),$$

for some regular C^1 function h. Taking into account the positivity $\overline{u}_0>0$, it follows $\overline{u}(t)>0$ for all $t>0$. In the same way, we have $\underline{u}(t)>0$.
In order to see $\underline{u}(t)<\overline{u}(t)$, we proceed by contradiction. If $\underline{u}(t)<\overline{u}(t)$ was false, then it would exist some $0<t_0<T_{max}$, such that

$$\underline{u}(t_0)=\overline{u}(t_0), \qquad \underline{u}(t)<\overline{u}(t), \quad \text{if} \quad t<t_0. \tag{3.26}$$

The solution of (3.23) with initial data $\underline{u}(t_0)=\overline{u}(t_0)$ satisfies $\underline{u}(t)=\overline{u}(t)$ for all $t>t_0$. We may extend the solution to the interval $(t_0-\epsilon,t_0)$, to obtain that $\underline{u}(t)=\overline{u}(t)$ in $(t_0-\epsilon,t_0+\epsilon)$, which contradicts (3.26) and proves

$$\underline{u}(t)<\overline{u}(t) \qquad t\in(0,T_{max}) \tag{3.27}$$

Since $\max\{\overline{u}_0,1+\|\overline{f}(t)\|_{L^\infty}\}$ is a supersolution of the first equation in (3.23) and $\underline{u}(t)=0$ is a subsolution of the second equation of (3.23), due to the uniqueness of solutions, it follows

$$0<\underline{u}(t) \qquad \text{and} \qquad \overline{u}(t)\le \max\{\overline{u}_0,1+\|\overline{f}(t)\|_{L^\infty}\}. \tag{3.28}$$

Relations (3.27) and (3.28) show that $T_{max}=\infty$ which ends the proof. □

Lemma 3.2. *Under assumptions (1.6) and (1.13), i.e.,*

$$\int_0^\infty |\overline{f}(t)-\underline{f}(t)|dt\le C<\infty, \qquad \textit{and} \qquad 2\chi<\mu,$$

there exists a positive constant K*, such that,*

$$\overline{u}(t)\le K\underline{u}(t). \tag{3.29}$$

Proof. Dividing the first equation in (3.23) by $\overline{u}(t)$ and the second one by $\underline{u}(t)$ we have

$$\frac{\overline{u}_t(t)}{\overline{u}(t)}=\chi(\overline{u}(t)-\underline{u}(t))+\mu(1-\overline{u}(t)+\overline{f}(t)), \tag{3.30}$$

$$\frac{\underline{u}_t(t)}{\underline{u}(t)}=\chi(\underline{u}(t)-\overline{u}(t))+\mu(1-\underline{u}(t)+\underline{f}(t)). \tag{3.31}$$

By subtracting (3.30) and (3.31), it results:

$$\frac{\overline{u}_t(t)}{\overline{u}(t)}-\frac{\underline{u}_t(t)}{\underline{u}(t)}=(2\chi-\mu)(\overline{u}(t)-\underline{u}(t))+\mu(\overline{f}(t)-\underline{f}(t)),$$

which is equivalent to

$$\frac{d}{dt}\ln\frac{\overline{u}(t)}{\underline{u}(t)}=(2\chi-\mu)(\overline{u}(t)-\underline{u}(t))+\mu(\overline{f}(t)-\underline{f}(t)). \tag{3.32}$$

Note that thanks to assumption (1.13) and Lemma 3.1 we get

$$\frac{d}{dt}\ln\frac{\overline{u}(t)}{\underline{u}(t)}\le\mu(\overline{f}(t)-\underline{f}(t)). \tag{3.33}$$

Integrating (3.33), we obtain

$$\ln\frac{\overline{u}(t)}{\underline{u}(t)}\le\mu\int_0^t|\overline{f}(s)-\underline{f}(s)|ds+\ln\frac{\overline{u}_0}{\underline{u}_0}.$$

Undoing the logarithm,

$$\frac{\overline{u}(t)}{\underline{u}(t)}\le\frac{\overline{u}_0}{\underline{u}_0}\exp\left(\int_0^t|\overline{f}(s)-\underline{f}(s)|ds\right)\le K,$$

where $K:=\overline{u}_0/\underline{u}_0\exp(C)$, C a positive constant which exists under assumption (1.6). □

Our next aim is to prove that the difference between $\overline{u}(t)$ and $\underline{u}(t)$ tend to 0 as t tends to infinity. We then state the following lemma:

Lemma 3.3. *If hypothesis (1.6), (1.7) and (1.13) are verified, i.e.,*

$$\int_0^\infty |\overline{f}(t) - \underline{f}(t)| dt \le C < \infty, \qquad \underline{f}(t) \ge -1 + \epsilon_1, \qquad \text{and} \qquad 2\chi - \mu < 0,$$

for initial data satisfying

$$0 < \underline{u}_0 < \overline{u}_0 < \infty,$$

the following chain of inequalities holds

$$0 < \epsilon_0 \le \underline{u}(t) \le \overline{u}(t) \le K_0, \qquad \text{for} \qquad t > 0$$

where

$$\epsilon_0 := \min\left\{\underline{u}_0, \frac{\mu\epsilon_1}{\chi K + \mu - \chi}\right\}$$

and

$$K_0 := \max\left\{\overline{u}_0, \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\chi/K + \mu - \chi}\right\}.$$

Proof. Thanks to Lemma 3.1, we have that $-\overline{u}(t) \ge -K\underline{u}(t)$ and (3.31) becomes

$$\frac{\underline{u}_t(t)}{\underline{u}(t)} \ge \chi(\underline{u}(t) - K\underline{u}(t)) + \mu(1 - \underline{u}(t) + \underline{f}(t)) = (\chi - \chi K - \mu)\underline{u}(t) + \mu(1 + \underline{f}(t)) \ge (\chi - \chi K - \mu)\underline{u}(t) + \mu\epsilon_1.$$

Hence,

$$\underline{u}(t) \ge \min\left\{\underline{u}_0, \frac{\mu\epsilon_1}{\chi K + \mu - \chi}\right\}.$$

In the same way, we show

$$\frac{\overline{u}_t(t)}{\overline{u}(t)} \le \chi(\overline{u}(t) - \frac{1}{K}\overline{u}(t)) + \mu(1 - \overline{u}(t) + \overline{f}(t)) = (\chi - \chi\frac{1}{K} - \mu)\overline{u}(t) + \mu(1 + \overline{f}(t)) \le (\chi - \chi\frac{1}{K} - \mu)\overline{u}(t) + 1 + \|f\|_{L^\infty(\Omega_\infty)}.$$

Then,

$$\overline{u}(t) \le \max\left\{\overline{u}_0, \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\chi/K + \mu - \chi}\right\}$$

and the proof ends. □

In the following lemma we apply assumption (1.8) to prove the asymptotic behavior of the solution of the auxiliary problem. We first notice that assumption (1.8) is equivalent to

$$\lim_{t\to\infty} \int_0^t e^{\epsilon(\tau - t)} \left(\sup_{x\in\Omega}\{f(x,\tau)\} - \inf_{x\in\Omega}\{f(x,\tau)\}\right) d\tau = 0 \tag{3.34}$$

for any $\epsilon > 0$.

Remark 3.1. *Under assumption (1.8) we have (3.34).*

Proof. We denote by $\overline{b}$ a non increasing function satisfying

$$\left(\sup_{x\in\Omega}\{f(\cdot,t\} - \inf_{x\in\Omega}\{f(\cdot,t)\}\right) \le \overline{b}(t) \qquad \text{and} \qquad \overline{b}(t) \to 0.$$

We have

$$\begin{aligned}
\int_0^t e^{\epsilon(\tau - t)} \left(\sup_{x\in\Omega}\{f(x,\tau)\} - \inf_{x\in\Omega}\{f(x,\tau)\}\right) d\tau &\le \int_0^t e^{\epsilon(\tau - t)} \left(\sup_{x\in\Omega}\{f(x,\tau)\} - \inf_{x\in\Omega}\{f(x,\tau)\}\right) d\tau \\
&\le \overline{b}(t/2)e^{-\epsilon t}\int_{t/2}^t e^{\epsilon\tau} d\tau + \int_0^{t/2} e^{\epsilon(\tau - t)}\overline{b}(\tau) d\tau \\
&\le \overline{b}(t/2)e^{-\epsilon t}\left(e^{\epsilon t} - e^{\epsilon t/2}\right) + t\overline{b}(0)e^{-\epsilon t/2} \to 0.
\end{aligned}$$

□

Lemma 3.4. *For every positive constants χ and μ as in (1.8) and every functions $\overline{f}$, $\underline{f}$ verifying (1.13), i.e.,*

$$2\chi < \mu \qquad \text{and} \qquad |\overline{f}(t) - \underline{f}(t)| \to 0, \qquad \text{when} \qquad t \to \infty,$$

the solutions of system (3.23) satisfy:

$$\frac{\overline{u}(t)}{\underline{u}(t)} \to 1, \qquad \text{when} \qquad t \to \infty. \tag{3.35}$$

Proof. We simplify expression (3.32) introducing $b(t)$ as

$$b(t) := \mu(\overline{f}(t) - \underline{f}(t)),$$

then (3.32) becomes

$$\frac{d}{dt} \ln \frac{\overline{u}(t)}{\underline{u}(t)} = (2\chi - \mu)\left(\frac{\overline{u}(t)}{\underline{u}(t)} - 1\right)\underline{u}(t) + b(t)$$

We define v given by $v(t) := \ln(\overline{u}(t)/\underline{u}(t))$ and substituting in the previous expression it results

$$\frac{d}{dt}v(t) = (2\chi - \mu)(e^{v(t)} - 1)\underline{u}(t) + b(t).$$

Thanks to Lemma 3.1 and 3.3, we obtain:

$$\frac{d}{dt}v(t) \leq \epsilon_0(2\chi - \mu)v(t) + b(t)$$

Now, taking $\epsilon_2 := (\mu - 2\chi)\epsilon_0$, the above inequality is reduced to

$$\frac{d}{dt}v(t) + \epsilon_2 v(t) \leq b(t).$$

After integration we get

$$v(t) \leq e^{-\epsilon_2 t}\left(\int_0^t b(s)e^{\epsilon_2 s}ds + v_0\right).$$

Using now Remark 3.1 we conclude

$$v(t) \to 0, \qquad \text{as} \qquad t \to \infty.$$

Finally, undoing the change, we have

$$\ln \frac{\overline{u}(t)}{\underline{u}(t)} \to 0, \qquad \text{as} \qquad t \to \infty,$$

which implies (3.35). □

The importance of this intermediate result lies in obtaining $(\overline{u}(t) - \underline{u}(t)) \to 0$ when $t \to \infty$, which implies that any other function bounded between them will inherit their asymptotic behaviour.
Let us consider now $u^* = u^*(t)$, the solution of the initial value problem

$$u_t^*(t) = \mu u^*(t)(1 - u^*(t) + f^*(t)) \tag{3.36}$$

with

$$u^*(0) = u_0^* := \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} > 0, \tag{3.37}$$

and $f^*(t)$ is a periodic function of period T.

Lemma 3.5. *Let f^* be defined in (1.10) satisfying (1.11), and let u_0^* be a positive value given by (3.37). Then, the solution $u^*(t)$ of (3.36) with initial data u_0^* is computed explicitly*

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \displaystyle\int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} \tag{3.38}$$

and it is a periodic function.

Proof. Let us note that (3.36) is a Bernoulli differential equation for $n = 2$. Using the change $w(t) = (u^*(t))^{(1-n)} = (u^*(t))^{-1}$, equation (3.36) is equivalent to

$$w_t(t) + \mu(1 + f^*(t))w(t) = \mu, \tag{3.39}$$

with the integrating factor $e^{\mu(1+f^*(t))}$ we obtain an explicit expression for w,

$$w(t) = \frac{w_0 + \int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}{e^{\int_0^t \mu(1+f^*(s))ds}}.$$

We impose here the periodic condition on w (taking into account the expression of w, for u periodic), that is, $w_0 = w(0) = w(T)$,

$$w_0 = \frac{w_0 + \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}{e^{\int_0^T \mu(1+f^*(s))ds}}$$

which is equivalent to

$$w_0(e^{\int_0^T \mu(1+f^*(s))ds} - 1) = \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau.$$

Computating, we then have

$$w_0 = \frac{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}{e^{\int_0^T \mu(1+f^*(s))ds} - 1} > 0,$$

which is positive under the assumption

$$e^{\int_0^T \mu(1+f^*(s))ds} - 1 > 0.$$

Furthermore, it is necessary that $w'(t) = w'(t + T)$, i.e., $f^*(t)$ is a time periodic function of period T. Now, returning to the initial equation with $u_0^* = {w_0}^{-1}$, we obtain

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}$$

where

$$u_0^* = \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} > 0.$$

To check the periodicity of u^* we compute $u^*(T)$ which is given by

$$u^*(T) = \frac{u_0^* e^{\int_0^T \mu(1+f^*(s))ds}}{1 + u_0^* \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} = \frac{e^{\int_0^T \mu(1+f^*(s))ds}}{(u_0^*)^{-1} + \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} = \frac{e^{\int_0^T \mu(1+f^*(s))ds}(e^{\int_0^T \mu(1+f^*(s))ds} - 1)}{e^{\int_0^T \mu(1+f^*(s))ds} \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} = u_0^*$$

and thanks to (3.36) and periodicity of f^* we conclude the lemma. □

The following lemma will be used to prove that the solution $u^*(t)$ of (3.36) is bounded between $\underline{u}(t)$ and $\overline{u}(t)$, solutions of the system (3.23):

Lemma 3.6. *Let f^* be defined in (1.10) satisfying (1.11), and let u_0^* be a positive value (3.37) verifying*

$$\underline{u}_0 < u_0^* < \overline{u}_0,$$

then, the solution $u^*(t)$ of (3.36) fulfills

$$\underline{u}(t) \leq u^*(t) \leq \overline{u}(t). \tag{3.40}$$

Proof. We prove first $u^*(t) \leq \overline{u}(t)$. Subtracting (3.30) and (3.36), it is obtained:

$$\frac{d}{dt} \ln \frac{\overline{u}(t)}{u^*(t)} = \chi(\overline{u}(t) - \underline{u}(t)) + \mu(u^*(t) - \overline{u}(t) + \overline{f}(t) - f^*(t)). \tag{3.41}$$

Now, by assumption (1.11) and Lemma 3.1, i.e., $\overline{f}(t) - f^*(t) > 0$ and $\overline{u}(t) - \underline{u}(t) > 0$, we can find an upper bound for the right hand side term of the equation (3.41) such that

$$\frac{d}{dt} \ln \frac{\overline{u}(t)}{u^*(t)} \geq \mu(u^*(t) - \overline{u}(t)). \tag{3.42}$$

By the Mean Value Theorem, it results

$$\mu(u^*(t) - \overline{u}(t)) = \mu(e^{\ln u^*(t)} - e^{\ln \overline{u}(t)}) = \mu\xi^* (\ln u^*(t) - \ln \overline{u}(t)) = \mu\xi^* \left(\ln \frac{u^*(t)}{\overline{u}(t)}\right) = -\mu\xi^* \left(\ln \frac{\overline{u}(t)}{u^*(t)}\right)$$

where $0 < u^*(t) \leq \xi^* \leq \overline{u}(t)$ if $u^*(t) < \overline{u}(t)$ and $\epsilon_0 \leq \overline{u}(t) \leq \xi^* \leq u^*(t)$ otherwise. Returning to the equation (3.42), we obtain

$$\frac{d}{dt} \ln \frac{\overline{u}(t)}{u^*(t)} \geq -\mu\xi^* \left(\ln \frac{\overline{u}(t)}{u^*(t)} \right), \qquad \text{with} \qquad \ln \frac{\overline{u}_0}{u_0^*} > 0.$$

Applying now Gronwall's lemma, we deduce

$$\ln \frac{\overline{u}(t)}{u^*(t)} \geq \ln \frac{\overline{u}_0}{u_0^*} e^{-\mu \int_0^t \xi^*} > 0$$

which implies

$$\frac{\overline{u}(t)}{u^*(t)} \geq 1.$$

In the same way we get $\underline{u}(t) \leq u^*(t)$ which ends the proof. □

The following theorem gives a precise description of the asymptotic behaviour under certain assumptions.

Theorem 3.1. *Let $(\underline{u}(t), \overline{u}(t))$ be the solution of system (3.23) and $u^*(t)$ the solution of (3.36) for f^* defined in (1.10) and satisfying (1.11), then, the following limits hold*

$$|\overline{u}(t) - u^*(t)| \to 0 \qquad \text{and} \qquad |\underline{u}(t) - u^*(t)| \to 0, \tag{3.43}$$

as $t \to \infty$.

Proof. Using now the result obtained in Lemma 3.6 we claim

$$0 < \overline{u}(t) - \underline{u}(t) = |\overline{u}(t) - u^*(t)| + |u^*(t) - \underline{u}(t)| \leq |\overline{u}(t) - \underline{u}(t)| + |\overline{u}(t) - \underline{u}(t)| = 2|\overline{u}(t) - \underline{u}(t)| \to 0.$$

Lemma 3.4 ends the proof. □

4. Comparison Principle and Asymptotic Behavior of Solutions

The aim of this section is to find the relation between the solution $(\underline{u}(t), \overline{u}(t))$ of the ODE's system (3.23) and the solution (u, v) of the PDE's system (1.2). Under some order relation between initial condition, we prove that such order is preserved. Taking into account the results of the previous sections, i.e. the functions $\underline{u}(t)$ and $\overline{u}(t)$ converge to $u^*(t)$ as $t \to \infty$, where $u^*(t)$ is a the periodic function defined in (3.38), we bound the solution of (1.2) between $\underline{u}(t)$ (lower bound) and $\overline{u}(t)$ (upper bound) to obtain the same qualitative behavior than $\underline{u}(t)$ and $\overline{u}(t)$. The proof follows the *rectangle method* used in Pao [17] for reaction diffusion systems, see also Negreanu and Tello [16] where the method is applied to Parabolic-Elliptic systems with chemotactic terms and [13].
Since $u_0^* > 0$ it is possible to find positive numbers $\underline{u}_0$ and $\overline{u}_0$ satisfying assumption (1.5), such that the inequalities

$$0 < \underline{u}_0 < u_0^* < \overline{u}_0 \tag{4.44}$$

hold. The main result of this section is as follows

Theorem 4.1. *Let $u_0 \in L^\infty(\Omega)$ and ϵ_0 be a positive number such that, for all positive initial data $(\underline{u}_0, \overline{u}_0)$ verifying*

$$\epsilon_0 \leq \underline{u}_0 \leq u_0 \leq \overline{u}_0, \qquad \text{in } \Omega,$$

and (4.44) the solution (u, v) of (1.2) is bounded and satisfies

$$\underline{u}(t) \leq u(x,t) \leq \overline{u}(t), \qquad \underline{u}(t) \leq v(x,t) \leq \overline{u}(t), \qquad (x,t) \in \Omega \times (0, \infty).$$

Let us introduce the following notations, in order to prove it:

$$\begin{aligned} \overline{U}(x,t) &:= u(x,t) - \overline{u}(t), & \underline{U}(x,t) &:= u(x,t) - \underline{u}(t) \\ \overline{V}(x,t) &:= v(x,t) - \overline{u}(t), & \underline{V}(x,t) &:= v(x,t) - \underline{u}(t) \end{aligned} \tag{4.45}$$

where (u, v) is the solution of (1.2) and $(\underline{u}(t), \overline{u}(t))$ is the solution of (3.23). We consider the standard positive and negative part functions defined by

$$(s)_+ = \begin{cases} s & \text{if } s \leq 0, \\ 0 & \text{in other case,} \end{cases} \qquad (s)_- = (-s)_+.$$

Now, our purpose is to prove that the positive and negative parts $\overline{U}_+$ and $\underline{U}_-$ are 0. To obtain the partial differential equation which is satisfied by $\overline{U}$, we subtract (1.2) and the first equation in (3.23) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}(u-\overline{u}(t))-\Delta(u-\overline{u}(t))=&-\chi\nabla(u-\overline{u}(t))\nabla v+\chi(u^2-\overline{u}(t)^2-uv+\overline{u}(t)\underline{u}(t))\\&+g(u)-g(\overline{u}(t))+\mu(fu-\overline{f}(t)\overline{u}(t)),\end{aligned}$$

where $g(u):=\mu u(1-u)$. Now, substituting $\overline{U}$, and adding $\pm\chi\overline{u}(t)v$ and $\pm\mu\overline{f}(t)u$

$$\begin{aligned}\overline{U}_t-\Delta\overline{U}=&-\chi\nabla\overline{U}\nabla v+\chi\overline{U}(u-v+\overline{u})+\chi\overline{u}(t)(\underline{u}(t)-v)\\&+\mu u(f-\overline{f}(t))+\mu\overline{f}(t)\,\overline{U}+g(u)-g(\overline{u}(t)).\end{aligned}$$

Applying the Mean Value Theorem to g for some $\xi(x,t)\in(u(x,t),\overline{u}(t))$ if $u\le\overline{u}(t)$ and $\xi(x,t)\in(\overline{u}(t),u(x,t))$ otherwise, the previous equation becomes

$$\begin{aligned}\overline{U}_t-\Delta\overline{U}=&-\chi\nabla\overline{U}\nabla v+\overline{U}[\chi(u-v+\overline{u}(t))+g'(\xi)]+\chi\overline{u}(t)(\underline{u}(t)-v)\\&+\mu u(f-\overline{f}(t))+\mu\overline{f}(t)\,\overline{U}.\end{aligned}\tag{4.46}$$

We multiply now by the test function $\overline{U}_+$, integrate by parts over Ω. Hence, after some rutinary computations, (4.46) remains

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega\overline{U}_+^2dx+\int_\Omega|\nabla\overline{U}_+|^2dx=-\chi\int_\Omega\overline{U}_+\nabla\overline{U}\nabla v\,dx+\chi\int_\Omega\overline{U}_+\overline{u}(\underline{u}(t)-v)dx\\&+\int_\Omega(\chi(u-v+\overline{u}(t))+g'(\xi))\overline{U}_+^2dx+\mu\int_\Omega\overline{U}_+u(f-\overline{f}(t))dx+\mu\int_\Omega\overline{f}(t)\,\overline{U}_+^2dx.\end{aligned}$$

We notice that

$$-\chi\int_\Omega\overline{U}_+\nabla\overline{U}\nabla v\,dx=-\frac{1}{2}\chi\int_\Omega\nabla\overline{U}_+^2\nabla v\,dx=\frac{\chi}{2}\int_\Omega\overline{U}_+^2\Delta v\,dx=\frac{\chi}{2}\int_\Omega\overline{U}_+^2(v-u)dx.$$

By the definition of $\overline{f}$, the term $\mu\int_\Omega\overline{U}_+u(f-\overline{f})dx<0$, so we find the bound

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega(\overline{U}_+)^2dx+\int_\Omega|\nabla(\overline{U}_+)|^2dx\le&\int_\Omega\left(\frac{\chi}{2}(u-v+2\overline{u}(t))+g'(\xi)\right)\overline{U}_+^2dx\\&+\chi\int_\Omega\overline{U}_+\overline{u}(t)(\underline{u}(t)-v)dx+\mu\int_\Omega\overline{f}(t)\,\overline{U}_+^2dx.\end{aligned}\tag{4.47}$$

To conclude the proof of Teorema 4.1, we need the following technical lemma before (see for instance Lemma 3.2 in [14]).

Lemma 4.1. *Let $\Omega\in\mathbb{R}^n$ an open and bounded set with regular boundary, then, for $q\ge 2$, and $h\in L^q(\Omega)$, the solution w of the problem*

$$\begin{cases}-\Delta w+w=h, & \text{in }\ \Omega,\\ \dfrac{\partial w}{\partial n}=0, & \text{in }\ \partial\Omega\end{cases}\tag{4.48}$$

satisfies

$$\|(w-c)_+\|_{L^q(\Omega)}\le C(\Omega,q)\|(h-c)_+\|_{L^q(\Omega)}$$

and

$$\|(w-c)_-\|_{L^q(\Omega)}\le C(\Omega,p,q)\|(f-c)_-\|_{L^q(\Omega)}$$

for any $c\in\mathbb{R}$.

Proof. To prove the first inequality of the lemma, we write equation (4.48) in the following way

$$-\Delta(w-c)+(w-c)=h-c,\qquad\text{in}\quad\Omega$$

and multiply by $(w-c)_+^{q-1}$, integrating by parts we get

$$(q-1)\int_\Omega(w-c)_+^{q-2}|\nabla(w-c)_+|^2+\int_\Omega(w-c)_+^q=\int_\Omega(w-c)_+^{q-1}(h-c).$$

Then,

$$(q-1)\int_\Omega(w-c)_+^{q-2}|\nabla(w-c)_+|^2+\int_\Omega(w-c)_+^q\le\int_\Omega(w-c)_+^{q-1}(h-c)_+$$

$$\leq \frac{q-1}{q}\int_\Omega (w-c)_+^{q-1} + \frac{1}{q}\int_\Omega (h-c)_+^q.$$

Then,

$$(q-1)\int_\Omega (w-c)_+^{q-2}|\nabla (w-c)_+|^2 + \frac{1}{q}\int_\Omega (w-c)_+^q \leq \frac{1}{q}\int_\Omega (h-c)_+^q \tag{4.49}$$

which implies

$$\int_\Omega (w-c)_+^q \leq \int_\Omega (h-c)_+^q$$

which proves the first inequality. In the same way we obtain the second inequality. □

Remark 4.1. *As a consequence of (4.49), if $q > n$ we have that*

$$\|(w-c)_+^{\frac{q}{2}}\|_{H^1(\Omega)} \leq C(q)\|(f-c)_+^{\frac{q}{2}}\|_{L^2(\Omega)}$$

thanks to Sobolev embedding, it results

$$\|(w-c)_+\|_{L^p(\Omega)} \leq C(q,p,\Omega)\|(f-c)_+\|_{L^q(\Omega)}$$

for any $p \in [q,\infty]$.

Remark 4.2. *As a consequence of the previous lemma, we have*

$$\|\overline{V}_+\|_{L^2(\Omega)} \leq \|\overline{U}_+\|_{L^2(\Omega)}$$

and

$$\|\underline{V}_-\|_{L^2(\Omega)} \leq \|\underline{U}_-\|_{L^2(\Omega)}$$

where $\underline{U}$, $\overline{U}$, $\underline{V}$ and $\overline{V}$ have been defined in (4.45).

We are able now to continue the proof of Theorem 4.1.

End of the proof of Theorem 4.1.

We notice that

$$\frac{\chi}{2}(u - v + 2\overline{u}(t)) + g'(\xi) \leq K_1 < \infty,$$

and

$$\chi\int_\Omega \overline{u}(t)(\underline{u}(t) - v)\overline{U}_+ dx \leq \chi\overline{u}_0\int_\Omega (\underline{u}(t)-v)_+\overline{U}_+ dx.$$

We can bound the terms of (4.47) as follows

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (\overline{U}_+)^2 dx + \int_\Omega |\nabla(\overline{U}_+)|^2 dxdx \quad &\leq K_1\int_\Omega \overline{U}_+^2 dx + \chi\overline{u}_0\int_\Omega (\underline{u}(t)-v)_+\overline{U}_+ dx + \mu||\overline{f}||_{L^\infty}\int_\Omega \overline{U}_+^2 dx \\ &\leq \left(K_1 + \mu||\overline{f}||_{L^\infty} + \frac{\chi\overline{u}_0}{2}\right)\int_\Omega \overline{U}_+^2 dx + \frac{\chi\overline{u}_0}{2}\int_\Omega (\underline{u}(t)-v)_+^2 dx.\end{aligned}$$

Now, thanks to Remark 4.2, it results

$$\frac{\partial}{\partial t}\int_\Omega (\overline{U}_+)^2 dx + \int_\Omega |\nabla(\overline{U}_+)|^2 dx \leq C_+\left(\int_\Omega \overline{U}_+^2 dx + \int_\Omega \overline{U}_-^2 dx\right), \qquad C_+ > 0.$$

In the same way we get

$$\frac{\partial}{\partial t}\int_\Omega (\underline{U}_-)^2 dx + \int_\Omega |\nabla(\underline{U}_-)|^2 dx \leq C_-\left(\int_\Omega \underline{U}_-^2 dx + \int_\Omega \underline{U}_+^2 dx\right), \qquad C_- > 0.$$

Adding both last equations, we have that

$$\frac{d}{dt}\left(\int_\Omega (\overline{U}_+^2 + \underline{U}_-^2)\right)dx \leq (C_+ + C_-)\int_\Omega \left(\overline{U}_+^2 + \underline{U}_-^2\right)dx.$$

Applying Gronwall's lemma, we obtain that $\overline{U}_+ = \underline{U}_- = 0$ (for more details, see for instance, [12] or [15]). Hence we have $\underline{u}(t) \leq u(x,t) \leq \overline{u}(t)$, for all $(x,t) \in \Omega\times(0,\infty)$. By Remark 4.2, we also obtain $\overline{V}_+ = \underline{V}_- = 0$ and therefore $\underline{u}(t) \leq v \leq \overline{u}(t)$ and the proof of the theorem ends. □

The following theorem gives a precise description of the asymptotic behavior of (u,v) under certain assumptions, using the bounds obtained. That is to say, the solution (u,v) behaves in the limit like $u^*(t)$, whose expression and periodic behavior we already know.

Theorem 4.2. *For any nonnegative initial data $u_0 > 0$, $u_0 \in L^\infty(\Omega)$ the solution (u, v) of (1.2) fulfills*

$$\lim_{t\to\infty} \|u(x,t) - u^*(t)\|_{L^\infty(\Omega)} + \|v(x,t) - u^*(t)\|_{L^\infty(\Omega)} = 0,$$

where $u^*(t)$ is given by (3.38).

Proof. The proof is immediate by Theorem 3.1 and Theorem 4.1. □

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