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# Global existence and asymptotic behavior of solutions to a Predator–Prey chemotaxis system with two chemicals

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ABSTRACT

In this article we consider the dynamics of two biological species "$u_1$" and "$u_2$", secreting the chemical substances "$v_2$" and "$v_1$", respectively. The biological species present the ability to orientate their movement towards the concentration of the chemical secreted by the other species. The problem is presented as a coupled system of four PDEs with chemotactic terms, two parabolic equations describing the evolution of the biological species "$u_i$" (for $i = 1, 2$) and two elliptic equations for the concentration of the chemicals "$v_i$" (for $i = 1, 2$). The source terms describe a Predator–Prey interaction, where the population of the predator "$u_2$" has a negative effect in the density of the prey "$u_1$" which affects positively in the population of the prey, its movement is oriented towards the higher concentration of the chemical secreted by the prey. The prey presents chemo-repulsion capabilities to avoid the predator throwout the chemical secreted by the predator "$v_1$". The system is presented as follows

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \cdot \nabla v_1) + g_1(u_1, u_2), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \cdot \nabla v_2) + g_2(u_1, u_2), & x \in \Omega, \quad t > 0, \\ -d_{v_1} \Delta v_1 + \alpha_1 v_1 = \beta_1 u_2, & x \in \Omega, \quad t > 0, \\ -d_{v_2} \Delta v_2 + \alpha_2 v_2 = \beta_2 u_1, & x \in \Omega, \quad t > 0, \end{cases}$$

where $\Omega$ is a bounded, open regular domain of $\mathbb{R}^n$, for $n \geq 1$ with regular boundary and $g_i(u_1, u_2)$ $(i = 1, 2)$ are the classical logistic terms of Lotka Volterra prey predator system. Under some conditions on the periodicity of the coefficients of the system we obtain the global existence and boundedness of classical solutions with given nonnegative initial function $u_i(x, 0) = u_i^0(x)$. Moreover, if $g_i(u_1, u_2)$ $(i = 1, 2)$ verify some further assumptions, the solutions of our system stabilize to some periodic functions in the large time limit, given coexistence of solutions for a range of parameters.



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## 1. Introduction

Predator–Prey systems of differential equations have been studied in the last hundred years by different authors, from the pioneering works of Lotka 1925 and Volterra in 1926 where the evolution of the species is given in terms of a system of two ODEs. The problem becomes a PDEs system of two parabolic equations when spatial diffusion of the species is considered. The mathematical model has been used in ecology to describe the spatial effects in the evolution of ecosystems. The system consists of two parabolic equations for the species $u_1$ and $u_2$, in $\Omega_\infty := \Omega \times (0, \infty)$:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1(t)\Delta u_1 + g_1(x, t, u_1, u_2), \\ \dfrac{\partial u_2}{\partial t} = d_2(t)\Delta u_2 + g_2(x, t, u_1, u_2), \end{cases} \tag{1.1}$$

where $g_1$ and $g_2$ are given functions describing the interaction of the species and $\Omega$ is a bounded, open and regular domain in $\mathbb{R}^n$, for $n \geq 1$ with smooth boundary. The system has been already studied from a mathematical point of view in the 80′s for a large range of interactions $g_i$, see for instance Pao [28], where the stability of (1.1) is obtained for a competitive case with

$$g_1(u_1, u_2) = u_1\left(a_{01} - a_{11}u_1 - a_{12}u_2\right), \quad g_2(u_1, u_2) = u_2\left(a_{02} - a_{21}u_1 - a_{22}u_2\right),$$

for constant coefficients $(a_{ij})_{i,j}$ for $i = 0, 1, 2$ and $j = 1, 2$.

Periodic situations frequently appear in populations dynamics, where cells, climatic or reproductive cycles among others, influence the amount of resources of the environment. Consequently, the parameters describing such influence should present some kind of periodicity in its asymptotic behavior.

In Cosner and Lazer [7], system (1.1) is studied for periodic in time coefficients $a_{01}$ and $a_{02}$ satisfying the Gopalsamy condition (see Gopalsamy [10]). In Ahmad and Lazer [1], the results of [7] are extended to the periodic in time dependence of all the coefficients $a_{ij}$ (for $i = 0, 1, 2$ and $j = 1, 2$). In Fu and Ruyun [9], the existence of periodic solutions is given for periodic in time and space dependent coefficients. The previous results have been generalized to almost periodic functions in Hetzer and Shen [12] for one and also to several species, see also [11].

Nevertheless, there exist common examples in nature, where the biological species movement is oriented by chemicals gradients, "Chemotaxis/Chemorepulsion" where predator moves towards the prey. Different types of situations may occur depending of the ability of the predator and the prey to orient their movement towards these chemical gradients.

- The predator may orient its movement towards the higher concentration of the chemical secreted by the prey or away from it, if it has the ability to orient its movement.
- The prey moves away from the higher concentration of the predator (chemorepulsion) or in the opposite direction, if it has the ability to do it.

Up to 9 different possible situations can take place and are considered in the article for a range of parameters.

The problem is described by the following reaction–diffusion system, where the populations densities $u_1$, $u_2$ follow a chemical gradient of two substances $v_1$, $v_2$ respectively,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1(t)\Delta u_1 - \chi_1 \nabla \cdot (u_1 \cdot \nabla v_1) + g_1(x, t, u_1, u_2), \\ \dfrac{\partial u_2}{\partial t} = d_2(t)\Delta u_2 - \chi_2 \nabla \cdot (u_2 \cdot \nabla v_2) + g_2(x, t, u_1, u_2), \\ -d_{v_1}\Delta v_1 + \alpha_1 v_1 = \beta_1 u_2, \\ -d_{v_2}\Delta v_2 + \alpha_2 v_2 = \beta_2 u_1, \end{cases} \tag{1.2}$$

$d_i = d_i(t)$ are smooth strictly positive $T$-periodic functions, $d_{v_i} > 0$, are the diffusion coefficients for the species $v_1$ and $v_2$, $\chi_i$ are the chemorepulsion coefficients, $\alpha_i > 0$, $\beta_i > 0$, for $i = 1, 2$. We assume Neumann boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v_1}{\partial \nu} = \frac{\partial v_2}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{1.3}$$

and bounded initial data

$$u_1(0, x) = u_1^0(x), \qquad u_2(0, x) = u_2^0(x), \quad x \in \Omega, \tag{1.4}$$

satisfying

$$u_i^0(x) \in C^{2+\alpha}(\Omega), \; \frac{\partial u_i^0}{\partial \nu} = 0 \;\text{ in } \partial\Omega, \; \underline{u}_i^0 \le u_i^0(x) \le \overline{u}_i^0 < \infty, \; x \in \Omega, \tag{1.5}$$

for some $\alpha > 0$ and $\overline{u}_i^0 \ge \underline{u}_i^0 > 0$, for $i = 1, 2$. Functions $g_i$, for $i = 1, 2$ have the expressions

$$g_1(u_1, u_2) = u_1 \left(a_{01}(x, t) - a_{11}(x, t)u_1 - a_{12}(x, t)u_2\right), \tag{1.6}$$

$$g_2(u_1, u_2) = u_2 \left(-a_{02}(x, t) + a_{21}(x, t)u_1 - a_{22}(x, t)u_2\right), \tag{1.7}$$

where the coefficients $a_{ij} = a_{i,j}(x, t)$ (for $i = 0, 1, 2$ and $j = 1, 2$) are smooth functions in $C^{\alpha, \frac{\alpha}{2}}(\Omega_\infty)$ describing the resources of the system and present a periodic asymptotic behavior in the sense

$$\lim_{t\to\infty} \sup_{x\in\Omega} |a_{i,j}(t, x) - a_{i,j}^*(t)| = 0,$$

where $a_{i,j}^* = a_{i,j}^*(t)$ are independent of $x$ and $T$-periodic in time.

Systems of two biological species with kinetic interaction have been considered in the last decade, see for instance Tello and Winkler [34], where the stability of homogeneous steady states is obtained for one chemical (see also Stinner, Tello and Winkler [30], Bai and Winkler [3] and Black, Lankeit and Mizukami [6]).

Predator–prey models of indirect taxis describe the evolution of the ecosystem for predators following a chemical secreted by the prey which doesn't present chemotactic ability, see Tello and Wrzosek [35], Li and Tao [20], Kentauro and Semba [19], and Tao and Winkler [32] among others.

Competitive systems of two biological species and a chemical with non-constant coefficients have been presented in Issa and Shen [17], [18] where the authors establish sufficient conditions for the existence of solutions and its asymptotic dynamics. For the one species case with time and space dependence coefficients and growth term we refer to the reader to the article Issa and Shen [16].

Recently, systems of two biological species with chemotactic abilities have been studied. In Cruz, Negreanu and Tello [8], the competitive system is also studied for a general case

$$g_2 = u_2(a_{02} - a_{21}u_1 - a_{22}u_2)$$

for constant coefficients $a_{ij}$ and under the assumptions

$$2|\chi_1|\beta_1 + a_{12} < a_{22}, \qquad 2|\chi_2|\beta_2 + a_{21} < a_{11}.$$

In [8] the global existence and asymptotic behavior are obtained for positive and bounded initial data.

In Zhang, Liu and Yang [38], the system (1.2) is studied for constant coefficients in the competitive case, i.e.,

$$g_i = \mu_i u_i(1 - a_{ij}u_j - u_i), \qquad \text{for } i \neq j \quad i,j = 1,2$$

under assumption $\mu_1\mu_2 > \chi_1\chi_2$ for $\chi_i \geq 0$, for $a_{01} = a_{02} = a_{11} = a_{22} = 1$ and $a_{21}, a_{12} \in (0,1)$. The authors prove the global existence of solutions for any bounded and regular initial data. If, in addition, $\chi_i < a_i\mu_i$ the authors obtain the convergence to a positive constant steady state

$$\left(\frac{1-a_{12}}{1-a_{12}a_{21}}, \frac{1-a_{21}}{1-a_{12}a_{21}}, \frac{1-a_{21}}{1-a_{12}a_{21}}, \frac{1-a_{12}}{1-a_{12}a_{21}}\right).$$

Recently, Zhang [37] has studied the competitive case for constant coefficients satisfying

$$a_1 > 1 > a_2 > 0,$$

for $\mu_1\mu_2 > \chi_1\chi_2$ and $\chi_1 \leq a_1\mu_1$, $\chi_2 < \mu_2$.

Zheng, Mu and Hu [39], have studied the fully parabolic system for the case $g_1 = g_2 = 0$, i.e., the case where there is no competition between the species. The authors obtained the global existence of solutions and the coexistence of the species for $\chi_i \in (-1,1)$. The problem with four equations remains open for large $\chi_i$ which is not considered in the previous cited works. The particular case $g_1 = g_2 = 0$, produces "blow up phenomenon" for a range of initial data and parameters, see Tao and Winkler [31].

The fully parabolic problem is analyzed in Black [5] for constant coefficients in the cases:

1- weak competition, i.e., $a_1, a_2 \in (0,1)$;
2- partially strong competition, i.e., $a_2 < 1 < a_1$;
3- fully strong competition, i.e., $a_2, a_1 > 1$.

In [5] the global existence of solutions is proven for

$$\mu_i > \frac{7}{2}\chi_i^2.$$

Coexistence and extinction are studied for different parameters and initial data. See also Mizukami [21] and Mizukami and Yokota [22].

The mathematical model that we deal with in this article is connected to the systems considered in [8], [38], [37] and [39], where the coefficients $a_{ij}$ are positive constants modeling the competitive interaction between the species, while in (1.2) we have a predator–prey interaction. The system also extends the Predator–Prey models with indirect taxis (see [35]) to the case where the prey has the ability to orient its movement following a chemical gradient related to the predator. As much as we know, the predator–prey system with two chemicals with non-constant coefficients has not been considered before in the literature from a mathematical point of view.

A natural question is to quantify the influence of the periodicity in the asymptotic behavior of the coefficients in the behavior of the solutions. Under which assumptions of converge in $a_{ij}$ to periodic functions we obtain a periodic asymptotic behavior of the solution? This article is devoted to study these questions among other qualitative properties of the solutions.

Throughout the paper, we use the following notation

$$\overline{a}_{ij}(t) = \sup_{x\in\Omega}\{a_{ij}(x,t)\}; \qquad \underline{a}_{ij}(t) = \inf_{x\in\Omega}\{a_{ij}(x,t)\}, \tag{1.8}$$

unless specified otherwise. From now on, given a function $\phi : \mathbb{R} \to \mathbb{R}$, we use the notation $\phi^L = \inf\phi$, $\phi^M = \sup\phi$. We have

$$\inf_{t\in\mathbb{R}} \underline{a}_{ij}(t) = \min_{t\in[0,T]} \underline{a}_{ij}(t) = a_{ij}^L$$
$$\sup_{t\in\mathbb{R}} \overline{a}_{ij}(t) = \max_{t\in[0,T]} \overline{a}_{ij}(t) = a_{ij}^M$$

For convenience and technical reasons, we assume the following hypothesis:

**(H1)** Functions $a_{i,j} \in C_{x,t}^{\alpha,\frac{\alpha}{2}}(\Omega_T)$ for any $T < \infty$, $\alpha \in (0,1)$ and satisfy

$$\int_0^\infty e^{\epsilon t}|\overline{a}_{i,j}(x,t) - \underline{a}_{i,j}(x,t)|dt \le C < \infty; \tag{1.9}$$
$$|\overline{a}_{i,j}(x,t) - \underline{a}_{i,j}(x,t)| \to 0, \qquad \text{as} \quad t \to \infty \tag{1.10}$$

and

$$|a_{i,j}^M| = \|a_{i,j}\|_{L^\infty(\Omega_\infty)} < \infty, \tag{1.11}$$

for some $\epsilon > 0$.

**(H2)** There exist some periodic functions

$$a_{i,j}^* : \mathbb{R}_+ \to \mathbb{R}$$

such that

$$a_{ij}^L \le \underline{a}_{ij}(t) \le a_{i,j}^*(t) \le \overline{a}_{ij}(t) \le a_{ij}^M. \tag{1.12}$$

**(H3)**

$$2|\chi_1|\beta_1 + a_{12}^M - a_{22}^L < 0 \quad \text{and} \qquad 2|\chi_2|\beta_2 + a_{21}^M - a_{11}^L < 0. \tag{1.13}$$

**(H4)**

$$\inf \frac{a_{21}^*}{a_{02}^*} > \sup \frac{a_{11}^*}{a_{01}^*} \tag{1.14}$$
$$\inf \frac{a_{22}^*}{a_{12}^*} > \sup \frac{a_{21}^*}{a_{11}^*}. \tag{1.15}$$

Notice that, as a consequence of (1.13) we have that there exists $\epsilon_0 > 0$ such that

$$a_{11}^L a_{22}^L - |\chi_1|\beta_1(|\chi_2|\beta_2 + a_{21}^M) > \epsilon_0. \tag{1.16}$$

Moreover, from (1.10) and (1.12), $a_{i,j}^*$ are unique for $i,j = 1,2$.

In this article we study the existence of global in time solutions of (1.2) and their asymptotic behavior. The result is enclosed in the following theorem.

**Theorem 1.1.** *Under assumptions (1.9)–(1.16), for any nonnegative initial data $(u_1^0, u_2^0)$ verifying (1.5), there exists a unique solution to (1.2)–(1.4) satisfying*

$$u_i, v_i \in C_{x,t}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T), \qquad \textit{for } i = 1,2 \textit{ and any } T < \infty;$$

*moreover the solution $(u_1, u_2, v_1, v_2)$ of (1.2) fulfills*

$$\lim_{t\to\infty}\left[\|u_1 - u_1^*\|_{L^\infty(\Omega)} + \|v_1 - \frac{\beta_1}{\alpha_1}u_2^*\|_{L^\infty(\Omega)}\right] = 0$$

$$\lim_{t\to\infty}\left[\|u_2-u_2^*\|_{L^\infty(\Omega)}+\|v_2-\frac{\beta_2}{\alpha_2}u_1^*\|_{L^\infty(\Omega)}\right]=0$$

*where* $(u_1^*, u_2^*)$ *is the unique positive T-periodic solution of system*

$$\begin{cases}\tilde u_1' = \tilde u_1(a_{01}^* - a_{11}^*\tilde u_1 - a_{12}^*\tilde u_2),\\ \tilde u_2' = \tilde u_2(-a_{02}^* - a_{22}^*\tilde u_2 + a_{21}^*\tilde u_1),\end{cases}\tag{1.17}$$

*with initial data* $(\tilde u_1^0, \tilde u_2^0) = (u_1^0, u_2^0)$.

The results obtained in Theorem 1.1 are valid for particular case $\chi_1 = \chi_2 = 0$ where the solutions have the same asymptotic behavior that the ODE system (1.17). In that case, the results are already known, see for instance [15] and reference therein.

The article is organized as follows: in Section 2 we introduce an auxiliary system of ODEs and study the convergence of the solutions of the auxiliary system to a periodic in time state. In Section 3 we prove, by using a comparison method "Rectangle Method" (see Tello and Winkler [33]), that the solutions of the PDEs system converge to the same periodic in time solution.

## 2. Super- and sub-solutions. Qualitative properties

In this section, we construct super- and sub-solutions of the nonlinear system (1.2). Such sub- and super-solutions are obtained as solutions of an auxiliary system of ordinary differential equations and allow us to obtain the asymptotic behavior of the solutions of (1.2).

We introduce the parameter $\tilde\beta_i$ defined by $\tilde\beta_i := \beta_i/d_{v_i}$ and to simplify the notations, we drop the tilde. Our strategy consists in relating and comparing the solutions of the initial PDE's system (1.2) and the solutions of the ODE's system (2.1) in order to prove the asymptotic behavior of the solutions of (1.2). We take the auxiliary functions $(\overline u_1, \underline u_1, \overline u_2, \underline u_2) = (\overline u_1(t), \underline u_1(t), \overline u_2(t), \underline u_2(t))$ defined implicitly as the solutions of the initial value problem

$$\begin{cases}\overline u_1' = |\chi_1|\beta_1\overline u_1(\overline u_2-\underline u_2)+\overline u_1(\overline a_{01}-\underline a_{11}\overline u_1-\underline a_{12}\underline u_2),\\ \underline u_1' = |\chi_1|\beta_1\underline u_1(\underline u_2-\overline u_2)+\underline u_1(\underline a_{01}-\overline a_{11}\underline u_1-\overline a_{12}\overline u_2),\\ \overline u_2' = |\chi_2|\beta_2\overline u_2(\overline u_1-\underline u_1)+\overline u_2(-\underline a_{02}-\underline a_{22}\overline u_2+\overline a_{21}\overline u_1),\\ \underline u_2' = |\chi_2|\beta_2\underline u_2(\underline u_1-\overline u_1)+\underline u_2(-\overline a_{02}-\overline a_{22}\underline u_2+\underline a_{21}\underline u_1),\end{cases}\tag{2.1}$$

for $t \in (0, \infty)$, with initial data

$$\overline u_1(0)=\overline u_1^0;\quad \underline u_1(0)=\underline u_1^0;\quad \overline u_2(0)=\overline u_2^0;\quad \underline u_2(0)=\underline u_2^0;\tag{2.2}$$

satisfying

$$0<\underline u_1^0<\overline u_1^0<\infty,\qquad\qquad 0<\underline u_2^0<\overline u_2^0<\infty.\tag{2.3}$$

**Lemma 2.1.** *The solution of system (2.1), with initial data (2.2) verifying (2.3), exists locally and satisfies* $[C^1(0,T_{max})]^4$ *for* $T_{max}$ *defined as follows*

$$\limsup_{t\to T_{max}}(|\overline u_1(t)|+|\underline u_1(t)|+|\overline u_2(t)|+|\underline u_2(t)|+t=\infty.$$

*Moreover the solution satisfies*

$$0 < \underline{u}_1 < \overline{u}_1 \tag{2.4}$$

$$0 < \underline{u}_2 < \overline{u}_2 \tag{2.5}$$

*for* $t \in (0, T_{max})$.

**Proof.** Notice that, since the right hand side terms of (2.1) is a second order polynomial, standard ODE theory gives the local existence of solutions of the system (2.1)–(2.2) in $(0, T_{max})$. The regularity is a consequence of the continuity of the coefficients. Therefore the solution exists as far as it remains bounded. We check that $\underline{u}_i > 0$, for $i = 1, 2$. We can write the corresponding equations of $\underline{u}_i$ from system (2.1) as $\underline{u}_i' = \underline{u}_i f_i(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$, $f_i$ being smooth functions. One proves easily that $\underline{u}_i = 0$ is a solution of the previous equation. By existence and uniqueness of the solution and taking into account that the initial data $\underline{u}_i^0$ is positive, it follows that $\underline{u}_i(t) > 0$ for all $t > 0$. To obtain $\underline{u}_i < \overline{u}_i$ we proceed by contradiction. Since the proof is similar to the case where the coefficients are constant, those we omit the details (see Negreanu–Tello [24] for details). □

**Lemma 2.2.** *The solution of system (2.1), with initial data (2.2) verifying (2.3), satisfies*

$$\overline{u}_i \le C_i < \infty \tag{2.6}$$

$$0 < \epsilon_1 e^{-\gamma t} < \underline{u}_i \tag{2.7}$$

*for* $i = 1, 2$, *any* $t \in (0, \infty)$, *some* $\epsilon_1 > 0$ *and* $\gamma > 0$.

**Proof.** We multiply the first equation in (2.1) by a positive constant $A$ and add third equation, to obtain

$$\begin{aligned}\frac{d}{dt}(A\overline{u}_1 + \overline{u}_2) &\le (A|\chi_1|\beta_1 + |\chi_2|\beta_2 + \overline{a}_{21})\overline{u}_1\overline{u}_2 - A\underline{a}_{11}\overline{u}_1^2 - \underline{a}_{22}\overline{u}_2^2 + A\overline{u}_1\overline{a}_{01} \\ &\le (A|\chi_1|\beta_1 + |\chi_2|\beta_2 + \overline{a}_{21})\overline{u}_1\overline{u}_2 - A\underline{a}_{11}\overline{u}_1^2 - \underline{a}_{22}\overline{u}_2^2 + \overline{a}_{01}(A\overline{u}_1 + \overline{u}_2).\end{aligned}$$

We take

$$A := \frac{2a_{11}^L a_{22}^L}{|\chi_1|^2 {\beta_1}^2} - \frac{|\chi_2|\beta_2 + a_{21}^M}{|\chi_1|\beta_1} > 0.$$

Notice that

$$\begin{aligned}&(A|\chi_1|\beta_1 + |\chi_2|\beta_2 + \overline{a}_{21})\overline{u}_1\overline{u}_2 - A\underline{a}_{11}\overline{u}_1^2 - \underline{a}_{22}\overline{u}_2^2 \\ &= \overline{u}_1^2\left[B\frac{\overline{u}_2}{\overline{u}_1} - \underline{a}_{22}\frac{\overline{u}_2^2}{\overline{u}_1^2} - A\underline{a}_{11}\right] \\ &= -\overline{u}_1^2\left[\left(\frac{\overline{u}_2}{\overline{u}_1} - \frac{B}{2\underline{a}_{22}}\right)^2 + \frac{4A\underline{a}_{22}\underline{a}_{11} - B^2}{4\underline{a}_{22}^2}\right]\end{aligned}$$

where

$$B := (A|\chi_1|\beta_1 + |\chi_2|\beta_2 + \overline{a}_{21}),$$

which defines

$$B_M := \sup B = (A|\chi_1|\beta_1 + |\chi_2|\beta_2 + \overline{a}_{21}^M).$$

Since

$$4A\underline{a}_{22}\underline{a}_{11} - B^2 = 4A\underline{a}_{22}\underline{a}_{11} - A^2(|\chi_1|\beta_1)^2 - (|\chi_2|\beta_2 + \overline{a}_{21})^2 - 2A|\chi_1|\beta_1(|\chi_2|\beta_2 + \overline{a}_{21}),$$

we replace in the above equation and thanks to (1.16) we obtain

$$4A\underline{a}_{22}\underline{a}_{11} - B^2 \geq 0.$$

Then

$$\frac{d}{dt}(A\overline{u}_1 + \overline{u}_2) \leq k_1(A\overline{u}_1 + \overline{u}_2) - k_2(A\overline{u}_1 + \overline{u}_2)^2$$

for

$$k_1 := a_{01}^M, \qquad k_2 := \frac{4Aa_{22}^L a_{11}^L - B_M^2}{(2a_{22}^M)^2},$$

which implies that

$$A\overline{u}_1 + \overline{u}_2 \leq C,$$

and proves (2.6). Thanks to Lemma 2.1,

$$\underline{u}_1 + \underline{u}_2 \leq C(1 + A^{-1}).$$

From (2.1) and the boundedness of $\overline{u}_1$ and $\overline{u}_2$ we deduce

$$\frac{d}{dt}\underline{u}_i \geq -c\underline{u}_i$$

which implies the wished result. □

Let $a_{ij}^*(t)$ be defined in (**H2**) and let $\tilde{u}_1(t), \tilde{u}_2(t)$ be the solutions of system (1.17), i.e.,

$$\begin{cases} \tilde{u}_1' = \tilde{u}_1(a_{01}^* - a_{11}^*\tilde{u}_1 - a_{12}^*\tilde{u}_2), \\ \tilde{u}_2' = \tilde{u}_2(-a_{02}^* - a_{22}^*\tilde{u}_2 + a_{21}^*\tilde{u}_1), \end{cases}$$

for $t \in (0, \infty)$, with initial data

$$\tilde{u}_1(0) = \tilde{u}_1^0; \quad \tilde{u}_2(0) = \tilde{u}_2^0. \tag{2.8}$$

Notice that as a consequence of the previous Lemmas, we have that $T_{max} = \infty$ which implies the global existence of the solutions.

System (1.17) models the interaction between a prey and a predator in a $T$ periodic environment. There are well known results about the existence and the stability of this class of solutions (see for instance [2], [36]). For readers' convenience we present the following lemma to be used in the proof of Theorem 1.1.

**Lemma 2.3.** *[A. Tineo [36], Z. Amine and R. Ortega [2]] Suppose that $a_{i,j}^*$ are T-periodic and*

$$\inf \frac{a_{21}^*}{a_{02}^*} > \sup \frac{a_{11}^*}{a_{01}^*}, \qquad \inf \frac{a_{22}^*}{a_{12}^*} > \sup \frac{a_{21}^*}{a_{11}^*}.$$

*Then, system (1.17) has a unique positive T-periodic solution* $(u_1^*, u_2^*)$ *such that*

$$(\tilde{u}_1(t) - u_1^*, \tilde{u}_2(t) - u_2^*) \to (0,0) \qquad as \quad t \to \infty, \tag{2.9}$$

*for any positive solution* $(\tilde{u}_1, \tilde{u}_2)$ *of (1.17).*

Throughout this section we prove that the two pairs of solutions of the ODE system (2.1), i.e., $(\overline{u}_1, \underline{u}_1)$ and $(\overline{u}_2, \underline{u}_2)$ have the same constant limits $u_1^*$ and $u_2^*$, respectively and, hence, also any function between them.

**Lemma 2.4.** *The pairs of solutions* $(\overline{u}_1, \overline{u}_2)$ *and* $(\underline{u}_1, \underline{u}_2)$ *are super- and sub-solutions of the prey–predator system (1.17) if the following relations between the initial data are satisfied*

$$0 < \underline{u}_1^0 < \tilde{u}_1^0 < \overline{u}_1^0, \qquad 0 < \underline{u}_2^0 < \tilde{u}_2^0 < \overline{u}_2^0. \tag{2.10}$$

*Thus, we have the ordering*

$$\begin{cases} \underline{u}_1(t) \le \tilde{u}_1(t) \le \overline{u}_1(t), \\ \underline{u}_2(t) \le \tilde{u}_2(t) \le \overline{u}_2(t). \end{cases} \tag{2.11}$$

**Proof.** Taking into account Lemma 2.1 and (1.8), we can rewrite system (2.1) as follows

$$\begin{cases} \overline{u}_1' \ge \overline{u}_1(a_{01}^* - a_{11}^*\overline{u}_1 - a_{12}^*\overline{u}_2), \\ \overline{u}_2' \ge \overline{u}_2(-a_{02}^* - a_{22}^*\overline{u}_2 + a_{21}^*\overline{u}_1) \end{cases} \tag{2.12}$$

and

$$\begin{cases} \underline{u}_1' \le \underline{u}_1(a_{01}^* - a_{11}^*\underline{u}_1 - a_{12}^*\underline{u}_2), \\ \underline{u}_2' \le \underline{u}_2(-a_{02}^* - a_{22}^*\underline{u}_2 + a_{21}^*\underline{u}_1), \end{cases} \tag{2.13}$$

for $t \in (0,\infty)$, with initial data (2.2)

$$\overline{u}_1(0) = \overline{u}_1^0; \quad \underline{u}_1(0) = \underline{u}_1^0; \quad \overline{u}_2(0) = \overline{u}_2^0, \quad \underline{u}_2(0) = \underline{u}_2^0;$$

satisfying (2.3)

$$0 < \underline{u}_1^0 < \overline{u}_1^0 \qquad 0 < \underline{u}_2^0 < \overline{u}_2^0,$$

for $t \in (0,\infty)$. By contradiction, we assume that there exists $t_0$, $t_0 \in (0,\infty)$ such that

$$\begin{cases} (\overline{u}_1(t_0) - \tilde{u}_1(t_0))(\overline{u}_2(t_0) - \tilde{u}_2(t_0))(\underline{u}_1(t_0) - \tilde{u}_1(t_0))(\underline{u}_2(t_0) - \tilde{u}_2(t_0)) = 0, \\ \overline{u}_i(t) > \tilde{u}_i(t), \qquad \underline{u}_i(t) < \tilde{u}_i(t), \text{ for } t < t_0 \text{ and } i = 1,2. \end{cases}$$

If $\overline{u}_2(t_0) - \tilde{u}_2(t_0) = 0$, then, after combination of (2.12) and (2.13) we get

$$\frac{d}{dt} \ln \frac{\overline{u}_2}{\tilde{u}_2} \ge -a_{22}^*(\overline{u}_2 - \tilde{u}_2), \text{ for } t \le t_0,$$

which implies, thanks to mean value theorem and positivity of $\tilde{u}_2$ in $(0, t_0]$ that

$$\ln \frac{\overline{u}_2(t_0)}{\tilde{u}_2(t_0)} > 0$$

and proves that $\overline{u}_2(t_0) > \tilde{u}_2(t_0)$.

In the same way we prove that $\underline{u}_2(t_0) < \tilde{u}_2(t_0)$.

To see that $\overline{u}_1(t_0) > \tilde{u}_1(t_0)$ we combine (2.12) and (2.13) to obtain

$$\frac{d}{dt}\left(\ln \frac{\overline{u}_1}{\tilde{u}_1}\right) \geq -a_{11}^*(\overline{u}_1 - \tilde{u}_1) - a_{12}^*(\underline{u}_2 - \tilde{u}_2) > -a_{11}^*(\overline{u}_1 - \tilde{u}_1)$$

Thanks to Lemma 2.3, assumption (**H3**), and positivity of $\tilde{u}_1(t_0)$ we have

$$\frac{d}{dt}\left(\ln \frac{\overline{u}_1}{\tilde{u}_1}\right) \geq -\epsilon_1 \left(\ln \frac{\overline{u}_1}{\tilde{u}_1}\right)$$

and after integration we get

$$\ln \frac{\overline{u}_1(t_0)}{\tilde{u}_1(t_0)} > 0$$

and proves that $\overline{u}_1(t_0) > \tilde{u}_1(t_0)$.

To end the proof, we obtain, thanks to 2.7) and mean value theorem that $\underline{u}_1(t_0) < \tilde{u}_1(t_0)$ as in the previous case. □

The asymptotic behavior of the solutions of the ODE system (2.12), is proven in the following theorem.

**Theorem 2.1.** *Let $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ be the solution of system (2.1) and $(u_1^*, u_2^*)$ the unique T-periodic solution of (1.17). Under assumptions (1.9)–(1.15) the below limits hold*

$$\lim_{t\to\infty} |\overline{u}_1(t) - u_1^*| \to 0, \qquad \lim_{t\to\infty} |\underline{u}_1(t) - u_1^*| \to 0, \qquad (2.14)$$
$$\lim_{t\to\infty} |\overline{u}_2(t) - u_2^*| \to 0, \qquad \lim_{t\to\infty} |\underline{u}_2(t) - u_2^*| \to 0.$$

In order to prove Theorem 2.1, we prove that $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ are actually global in time and bounded.

The importance of this result lies in obtaining $(\overline{u}_i - \underline{u}_i) \to 0$ when $t \to \infty$, $i = 1, 2$, which implies that any other function bounded between them will inherit their asymptotic behavior. The proof is given into several steps presented in the following lemmas:

**Lemma 2.5.** *Under hypothesis (1.9)–(1.11) and (1.13) we get*

$$\overline{u}_1 \leq M\underline{u}_1 \quad \textit{and} \quad \overline{u}_2 \leq M\underline{u}_2, \qquad (2.15)$$

*for a positive constant M.*

**Proof.** By dividing each equation that fulfills $\overline{u}_i$ and $\underline{u}_i$ of (2.1) by $\overline{u}_i$ and $\underline{u}_i$, respectively, and operating with the resulting equations, thanks to (1.13) and Lemma 2.1 we have

$$\frac{\overline{u}_1'}{\overline{u}_1} - \frac{\underline{u}_1'}{\underline{u}_1} + \frac{\overline{u}_2'}{\overline{u}_2} - \frac{\underline{u}_2'}{\underline{u}_2} \leq (\overline{a}_{01} - \underline{a}_{01}) + (\overline{a}_{02} - \underline{a}_{02})$$
$$+ C[(\overline{a}_{11} - \underline{a}_{11}) + (\overline{a}_{21} - \underline{a}_{21}) + (\overline{a}_{12} - \underline{a}_{12}) + (\overline{a}_{22} - \underline{a}_{22})]$$

for $C > 0$. By (1.9)–(1.11) and after integration

$$\ln \frac{\overline{u}_1}{\underline{u}_1} + \ln \frac{\overline{u}_2}{\underline{u}_2} \leq c_0 \tag{2.16}$$

for some positive constant $0 < c_0 < \infty$. This means that $\ln \frac{\overline{u}_1}{\underline{u}_1} \leq c_0$ so $\overline{u}_1 \leq M_1 \underline{u}_1$ with $M := e^{c_0}$, and the same holds for $\overline{u}_2$ and $\underline{u}_2$. □

**Lemma 2.6.** *Under hypothesis (1.9)–(1.13) we have*

$$\frac{\overline{u}_1}{\underline{u}_1} \to 1 \quad \textit{and} \quad \frac{\overline{u}_1}{\underline{u}_1} \to 1, \quad \textit{as} \quad t \to \infty. \tag{2.17}$$

**Proof.** Operating with the equation of (2.1) and proceeding as in the previous lemma, the following identity holds

$$\begin{aligned}
\frac{\overline{u}_1'}{\overline{u}_1} - \frac{\underline{u}_1'}{\underline{u}_1} + \frac{\overline{u}_2'}{\overline{u}_2} - \frac{\underline{u}_2'}{\underline{u}_2} &= (2|\chi_1|\beta_1 + \overline{a}_{12} - \underline{a}_{22})(\overline{u}_2 - \underline{u}_2) + (2|\chi_2|\beta_2 + \overline{a}_{21} - \underline{a}_{11}) + \\
&+ (\overline{u}_1 - \underline{u}_1) + (\overline{a}_{01} - \underline{a}_{01}) + (\overline{a}_{02} - \underline{a}_{02}) + \\
&+ \underline{u}_1[(\overline{a}_{11} - \underline{a}_{11}) + (\overline{a}_{21} - \underline{a}_{21})] + \underline{u}_2[(\overline{a}_{12} - \underline{a}_{12}) + (\overline{a}_{22} - \underline{a}_{22})].
\end{aligned}$$

Denoting by $A = A(t)$ the following term

$$A = (\overline{a}_{01} - \underline{a}_{01}) + (\overline{a}_{02} - \underline{a}_{02}) + \underline{u}_1[(\overline{a}_{11} - \underline{a}_{11}) + (\overline{a}_{21} - \underline{a}_{21})] + \underline{u}_2[(\overline{a}_{12} - \underline{a}_{12}) + (\overline{a}_{22} - \underline{a}_{22})]$$

we have that

$$\begin{aligned}
\frac{d}{dt}\left(\ln \frac{\overline{u}_1}{\underline{u}_1} + \ln \frac{\overline{u}_2}{\underline{u}_2}\right) &= (2|\chi_1|\beta_1 + \overline{a}_{12} - \underline{a}_{22})\left(\frac{\overline{u}_2}{\underline{u}_2} - 1\right)\underline{u}_2 + \\
&+ (2|\chi_2|\beta_2 + \overline{a}_{21} - \underline{a}_{11})\left(\frac{\overline{u}_1}{\underline{u}_1} - 1\right)\underline{u}_1 + A(t).
\end{aligned}$$

Following the notation $w_1 = \ln \frac{\overline{u}_1}{\underline{u}_1}$ and $w_2 = \ln \frac{\overline{u}_2}{\underline{u}_2}$, the above equations become

$$\begin{aligned}
\frac{d}{dt}(w_1 + w_2) &= (2|\chi_1|\beta_1 + \overline{a}_{12} - \underline{a}_{22})(e^{w_2} - 1)\underline{u}_2 + \\
&+ (2|\chi_2|\beta_2 + \overline{a}_{21} - \underline{a}_{11})(e^{w_1} - 1)\underline{u}_1 + A(t).
\end{aligned}$$

By means of the previous lemmas and (1.12), it follows that

$$\frac{d}{dt}(w_1 + w_2) \leq -\epsilon(w_1 + w_2) + A(t), \tag{2.18}$$

for some $\epsilon > 0$. Notice that existence of $\epsilon$ is guaranteed by Lemmas 2.4 and 2.5, positivity of $w_1$ and $w_2$ and assumptions (1.13).

We integrate now (2.18)

$$(w_1 + w_2)(t) \leq \int_0^t A(s)ds + w_1^0 + w_2^0 + \int_0^t -\epsilon(w_1 + w_2)(s)ds \tag{2.19}$$

and we use Gronwall's inequality to reach the result:

$$(w_1 + w_2)(t) \le e^{-\epsilon t}\left(\int_0^t e^{\epsilon s} A(s) ds + w_1^0 + w_2^0\right). \tag{2.20}$$

Taking limits as $t \to \infty$, using (1.9)–(1.10), we have that $w_1 + w_2 \to 0$ and then $\ln \frac{\overline{u}_1}{\underline{u}_1} + \ln \frac{\overline{u}_2}{\underline{u}_2}$ which implies $\frac{\overline{u}_i}{\underline{u}_i} \to 1$. Therefore (2.17) is satisfied and the proof ends. □

**End of the proof of Theorem 2.1.** Theorem 2.1 is a direct consequence of these properties of the solutions. Thanks to Lemmas 2.4, 2.6 and relation (2.9) we conclude the proof.

## 3. Comparison principle and asymptotic behavior of solutions

From now we have that the functions $(\underline{u}_1, \overline{u}_1, \underline{u}_2, \overline{u}_2) - (u_1^*, u_1^*, u_2^*, u_2^*)$ converge to 0 as $t \to \infty$, where $(u_1^*, u_1^*, u_2^*, u_2^*)$ are the unique $T$-periodic solution of (1.17), under different restrictions on the coefficients $a_{ij}$. We bound the solution of (1.2) between $\underline{u}_{1,2}$ (lower bound) and $\overline{u}_{1,2}$ (upper bound) to obtain the same qualitative behavior than $\underline{u}_{1,2}$ and $\overline{u}_{1,2}$. The proof follows the *rectangle method* used in Pao [29] for reaction diffusion systems, see also Negreanu and Tello [24], [25] where the method is applied to Parabolic–Elliptic systems with chemotactic terms.

Thanks to assumption (1.5), we have positive numbers $(\underline{u}_1^0, \overline{u}_1^0, \underline{u}_2^0, \overline{u}_2^0)$ such that

$$0 < \underline{u}_1^0 \le u_1^0(x) \le \overline{u}_1^0, \tag{3.1}$$

$$0 < \underline{u}_2^0 \le u_2^0(x) \le \overline{u}_2^0 \tag{3.2}$$

for all $x \in \Omega$.

Developing the second order terms in (1.2) we get

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 - \chi_1 \nabla u_1 \cdot \nabla v_1 + \chi_1 u_1 \left( \dfrac{\beta_1}{d_{v_1}} u_2 - \dfrac{\alpha_1}{d_{v_1}} v_1 \right) + g_1 \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 - \chi_2 \nabla u_2 \cdot \nabla v_2 + \chi_2 u_2 \left( \dfrac{\beta_2}{d_{v_2}} u_1 - \dfrac{\alpha_2}{d_{v_2}} v_2 \right) + g_2. \end{cases} \tag{3.3}$$

The main theorem of this section is as follows

**Theorem 3.1.** *Let $(u_1^0, u_2^0) \in (L^\infty(\Omega))^2$. The solution of (1.2)–(1.4) with initial data verifying (3.1), (3.2) is bounded and satisfies*

$$\underline{u}_1(t) \le u_1(t,x) \le \overline{u}_1(t), \quad \frac{\beta_2}{\alpha_2}\underline{u}_1(t) \le v_2(t,x) \le \frac{\beta_2}{\alpha_2}\overline{u}_1(t), \quad (x,t) \in \Omega \times (0,\infty), \tag{3.4}$$

$$\underline{u}_2(t) \le u_2(t,x) \le \overline{u}_2(t), \quad \frac{\beta_1}{\alpha_1}\underline{u}_2(t) \le v_1(t,x) \le \frac{\beta_1}{\alpha_1}\overline{u}_2(t), \quad (x,t) \in \Omega \times (0,\infty), \tag{3.5}$$

*where $(\underline{u}_i, \overline{u}_i)$ is the solution of the ODE system (2.1).*

Let us define the functions

$$\overline{U}_i(x,t) := u_i(t,x) - \overline{u}_i(t), \qquad \underline{U}_i(x,t) := u_i(t,x) - \underline{u}_i(t), \tag{3.6}$$

$$\overline{V}_i(x,t) := v_i(t,x) - \frac{\beta_i}{\alpha_i}\overline{u}_j(t), \qquad \underline{V}_i(x,t) := v_i(t,x) - \frac{\beta_i}{\alpha_i}\underline{u}_j(t), \tag{3.7}$$

$i, j = 1, 2$ with $i \neq j$ where $(u_1, u_2, v_1, v_2)$ and $(\underline{u}_1, \overline{u}_1, \underline{u}_2, \overline{u}_2)$ are the solutions of (1.2)–(1.4) and (2.1), respectively and we use from now on the usual notation for the positive part of a function $f$ given by $(f)_+ = f$ if $f \geq 0$, and 0 otherwise. The negative part function is defined by $(f)_- = (-f)_+$.

We aim to prove that the positive and negative parts $(\overline{U}_i)_+$, $(\underline{U}_i)_-$ are identically zero and therefore the solutions inherit the order $\underline{u}_i < u_i < \overline{u}_i$.

First, for an arbitrary $T > 0$ and due to the fact that the functions $(u_1, u_2, v_1, v_2)$ are continuous and differentiable in $\Omega \times (0, T)$ we can find $c_1(T) \geq 0$ such that

$$u_1(x,t) \leq c_1(T),\ u_2(x,t) \leq c_1(T),\ v_1(x,t) \leq c_1(T),\ v_2(x,t) \leq c_1(T). \tag{3.8}$$

**Proof.** First we deduce the equation that satisfy $\overline{U}_i$ and $\underline{U}_i$ subtracting equations in (1.2) and (2.1), respectively

$$\begin{aligned}
&\frac{\partial(u_1 - \overline{u}_1)}{\partial t} - d_1\Delta(u_1 - \overline{u}_1) = -\chi_1\nabla(u_1 - \overline{u}_1)\nabla v_1 - \chi_1\alpha_1 u_1 v_1 + u_1 a_{01}\\
&+\beta_1\left(\chi_1 u_1 u_2 - |\chi_1|\overline{u}_1\overline{u}_2 + |\chi_1|\overline{u}_1\underline{u}_2\right) - \overline{u}_1\overline{a}_{01} - u_1^2 a_{11}\\
&+\overline{u}_1^2\underline{a}_{11} - a_{12}u_1u_2 + \underline{a}_{12}\overline{u}_1\underline{u}_2.
\end{aligned}$$

Taking into account the definitions (1.8) for $\overline{a}_{ij}$ and $\underline{a}_{ij}$, we obtain

$$\begin{aligned}
&\frac{\partial(u_1 - \overline{u}_1)}{\partial t} - d_1\Delta(u_1 - \overline{u}_1)\\
&\leq -\chi_1\nabla(u_1 - \overline{u}_1)\nabla v_1 + \beta_1\left(\chi_1 u_1 u_2 - |\chi_1|\overline{u}_1\overline{u}_2 + |\chi_1|\overline{u}_1\underline{u}_2\right) - \chi_1\alpha_1 u_1 v_1\\
&+\overline{a}_{01}(u_1 - \overline{u}_1) - \underline{a}_{11}(u_1^2 - \overline{u}_1^2) - \underline{a}_{12}(u_1u_2 - \overline{u}_1\underline{u}_2)\\
&\leq -\chi_1\nabla(u_1 - \overline{u}_1)\nabla v_1 + \beta_1\left(\chi_1 u_1 u_2 - |\chi_1|\overline{u}_1\overline{u}_2 + |\chi_1|\overline{u}_1\underline{u}_2\right)\\
&-\underline{a}_{12}(u_1u_2 - \overline{u}_1\underline{u}_2) - \chi_1\alpha_1 u_1 v_1 + g(u_1) - g(\overline{u}_1),
\end{aligned}$$

where $g(u_1) := u_1(\overline{a}_{01} - \underline{a}_{11}u_1)$. For $\underline{U}_1$ we proceed in the same way. We multiply now the previous equation by the test function $(\overline{U}_1)_+$, operating and integrating by parts, we reach

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\overline{U}_1)_+^2 + d_1\int_\Omega|\nabla(\overline{U}_1)_+|^2 \leq k_1(T)\left(\int_\Omega(\overline{U}_1)_+^2 + \int_\Omega(\overline{U}_2)_+^2 + \int_\Omega(\underline{U}_2)_-^2\right)$$

for all $t \in (0, T)$. In the same fashion we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\underline{U}_1)_-^2 + d_1\int_\Omega|\nabla(\underline{U}_1)_-|^2 \leq k_2(T)\left(\int_\Omega(\underline{U}_1)_-^2 + \int_\Omega(\overline{U}_2)_+^2 + \int_\Omega(\underline{U}_2)_-^2\right),$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\overline{U}_2)_+^2 + d_2\int_\Omega|\nabla(\overline{U}_2)_+|^2 \leq k_3(T)\left(\int_\Omega(\overline{U}_2)_+^2 + \int_\Omega(\overline{U}_1)_+^2 + \int_\Omega(\underline{U}_1)_-^2\right),$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\underline{U}_2)_-^2 + d_2\int_\Omega|\nabla(\underline{U}_2)_-|^2 \leq k_4(T)\left(\int_\Omega(\underline{U}_2)_-^2 + \int_\Omega(\overline{U}_1)_+^2 + \int_\Omega(\underline{U}_1)_-^2\right),$$

for some positive constants $k_1(T)$, $k_2(T)$, $k_3(T)$ and $k_4(T)$. Now we add the above four equations to get

$$\frac{d}{dt}\left(\int_\Omega (\overline{U}_1)_+^2 + (\underline{U}_1)_-^2 + (\overline{U}_2)_+^2 + (\underline{U}_2)_-^2\right)$$
$$\leq k(T)\left(\int_\Omega (\overline{U}_1)_+^2 + (\underline{U}_1)_-^2 + (\overline{U}_2)_+^2 + (\underline{U}_2)_-^2\right),$$

with $k(T) = \max\{k_i(T) : i = 1...4\}$. It is possible now to apply Gronwall's lemma and recalling hypothesis (3.1), (3.2) we have $(\overline{U}_i^0)_+ = (\underline{U}_i^0)_- = 0$, (for $i = 1, 2$), so we conclude

$$\int_\Omega \left((\overline{U}_1)_+^2 + (\underline{U}_1)_-^2 + (\overline{U}_2)_+^2 + (\underline{U}_2)_-^2\right) = 0 \qquad \forall t \in (0,T) \tag{3.9}$$

and obtain that $\overline{U}_{i,+} = \underline{U}_{i,-} = 0$, $i = 1, 2$ (for more details, see for instance, [26] or [24]). Hence we have

$$\underline{u}_1(t) \leq u_1(t,x) \leq \overline{u}_1(t) \qquad \underline{u}_2(t) \leq u_2(t,x) \leq \overline{u}_2(t) \qquad (x,t) \in \Omega \times (0,T).$$

By Lemma 3.2 in [27] (see also [26], [23] for more details) we obtain $\overline{V}_{i,+} = \underline{V}_{i,-} = 0$, $i = 1, 2$ and therefore

$$\frac{\beta_2}{\alpha_2}\underline{u}_1(t) \leq v_2(t,x) \leq \frac{\beta_2}{\alpha_2}\overline{u}_1(t), \quad \frac{\beta_1}{\alpha_1}\underline{u}_2(t) \leq v_1(t,x) \leq \frac{\beta_1}{\alpha_1}\overline{u}_2(t),$$

for $(x,t) \in \Omega \times (0,T)$. As $T > 0$ is arbitrary, we take limits as $T \to \infty$ and the proof of the Theorem 3.1 ends. □

**Proof of Theorem 1.1.** The first part of the proof of the theorem is given in the following lemma.

**Lemma 3.1.** *Under assumptions (1.9)–(1.15), there exists a unique solution* $(u_1, u_2, v_1, v_2)$ *to (1.2)–(1.4) in* $(0, \infty)$ *satisfying*

$$u_i, v_i \in C_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_T), \qquad \text{for } i = 1, 2 \text{ and any } T < \infty.$$

*Moreover, for* $i = 1, 2$,

$$u_i(t,x) \geq 0, \quad v_i(x,t) \geq 0, \qquad x \in \Omega, \quad t < \infty. \tag{3.10}$$

**Proof.** We first consider $T_{max}$ such that

$$\limsup_{t \to T_{max}} \left(\|u_i(t)\|_{L^\infty(\Omega)} + \|v_i(t)\|_{L^\infty(\Omega)} + t\right) = \infty. \tag{3.11}$$

To obtain the local existence of solutions in $L^2(0 : T, H^1(\Omega)) \cap L^\infty(\Omega_T)$ for any $T < T_{max}$, we apply standard fixed point theory, see for instance Horstmann [13], Biler [4], Horstmann and Winkler [14], or Negreanu and Tello [25]. As a consequence of Theorem 3.1 and Lemmas 2.4 and 2.5 we conclude that $u_i$ are bounded functions. Thanks to maximum principle we get

$$\|v_i\|_{L^\infty(\Omega)} \leq \frac{\beta_i}{\alpha_i}\|u_j\|_{L^\infty(\Omega)}, \quad i \neq j \tag{3.12}$$

which implies that $T_{max} = \infty$.

The regularity of $u_i$ (for $i = 1, 2$) is a consequence of the parabolic and elliptic regularity of the equations, the regularity of the coefficients and the boundedness of $u_i$ and $v_i$. Uniqueness of solutions is obtained by contradiction, following standard arguments. □

The asymptotic behavior of the solutions is a consequence of Theorems 3.1 and Theorem 2.1. □

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