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On a parabolic–elliptic system with gradient dependent chemotactic coefficient [☆]

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Abstract

We consider a second order PDEs system of Parabolic–Elliptic type with chemotactic terms. The system describes the evolution of a biological species “ u ” moving towards a higher concentration of a chemical stimuli “ v ” in a bounded and open domain of \mathbb{R}^N . In the system considered, the chemotaxis sensitivity depends on the gradient of v , i.e., the chemotaxis term has the following expression

$$-div \left(\chi u |\nabla v|^{p-2} \nabla v \right),$$

where χ is a positive constant and p satisfies

$$p \in (1, \infty), \quad \text{if } N = 1 \quad \text{and} \quad p \in \left(1, \frac{N}{N-1} \right), \quad \text{if } N \geq 2.$$

We obtain uniform bounds in $L^\infty(\Omega)$ and the existence of global in time solutions. For the one-dimensional case we prove the existence of infinitely many non-constant steady-states for $p \in (1, 2)$ for any χ positive and a given positive mass.

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1. Introduction

Chemotaxis is the ability of some living organisms to orient their movement along a chemical concentration gradient. The process has been extensively studied from a biological point of view after the development of the microscope during the XIX century. In the last decades, several mathematical models have been presented to describe the phenomenon, after the pioneering works of Patlak [28] and Keller and Segel [22], [23] (see also the review articles Horstmann [19], [20] and Bellomo et al. [3] and references therein for more extensive literature in the subject). The original model in [22] describes the evolution of a biological species, denoted by “ u ” in terms of a parabolic equation, with linear diffusion and a second order nonlinear term in the form

$$-div(\chi u \nabla v),$$

where v denotes the concentration of the chemical stimuli.

In the last years, linear diffusion of the biological species “ u ” has been replaced in different ways:

- by nonlinear diffusion at the form $-div(\phi(u)\nabla u)$, see for instance Wrzosek [36], Cieslak and Morales-Rodrigo [16], Cieslak and Winkler [17], Winkler [35].
- by fractional diffusion, see J. Burczak, R. Granero-Belinchón [13] and [14] among others.
- by nonlinear diffusion depending on $|\nabla u|^p$, (p-laplacian), see Bendahmane [6].

The system has been also studied for several biological species, see for instance Tang and Tao [31], Tello and Winkler [33], Stinner, Tello and Winkler [30], Negreanu and Tello [26] and [27], Wang and Wu [34] among others. In the last years, several mathematical models have considered the chemotactic sensitivity coefficient “ χ ” dependent on ∇v instead of constant. For instance, in Bellomo and Winkler [4] and [5] (see also Bellomo et al. [2]), a chemotaxis system is analyzed for a chemotactic term of the form

$$-div\left(\frac{\chi u}{[1 + |\nabla v|^2]^{\frac{1}{2}}}\nabla v\right).$$

In Bianchi, Painter and Sherratt [7] the authors consider the term

$$-div\left(\frac{\chi u}{(1 + \omega u)}\frac{\nabla v}{(1 + \eta|\nabla v|)}\right),$$

for some positive constants χ , ω and η . In [7], the authors study a system of four PDEs of parabolic type in an one-dimensional spatial domain coupled with an ODE modeling Lymphangiogenesis in wound healing.

A general chemotactic term is presented as

$$-div[u\tilde{\chi}(u, v, |\nabla v|)\nabla v],$$

where $\tilde{\chi}$ is a continuous function for $|\nabla v| > 0$.

In the present work we consider a simplified case, where $\tilde{\chi}$ is given by

$$\tilde{\chi}(u, v, |\nabla v|) = \chi |\nabla v|^{p-2},$$

for some positive constant χ and $p > 1$.

We study the problem in a bounded spatial open domain $\Omega \subset \mathbb{R}^N$, with regular boundary $\partial\Omega$ and denote by \vec{n} the outward pointing normal vector on the boundary $\partial\Omega$. The equation for v is restricted to the elliptic case, for simplicity, we assume that v satisfies the Poisson equation and the system studied is the following

$$u_t - \Delta u = -div(\chi u |\nabla v|^{p-2} \nabla v), \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$-\Delta v = u - M, \quad x \in \Omega, \quad t > 0, \tag{1.2}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{1.3}$$

and a non-negative initial data

$$u(0, x) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

satisfying

$$\frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx = M.$$

Notice that if (u, v) is a solution, $(u, v + k)$ is also a solution for any constant k . To obtain uniqueness of the problem, we impose a given mass for the species v , i.e., we assume

$$\int_{\Omega} v = 0. \tag{1.5}$$

The problem for $p = 2$ has been already analyzed by different authors, starting with the work of Jäger and Luckhaus [21] and Biler [8], and [9], see also Nagai [24], Senba [29], Naito and Suzuki [25], Blanchet, Dolbeault and Perthame [11], Blanchet, Carrillo and Masmoundi [10] among others.

In this article we analyze the case where p satisfies

$$p \in (1, \infty), \quad \text{if } N = 1 \quad \text{and} \quad p \in \left(1, \frac{N}{N-1}\right), \quad \text{if } N \geq 2, \tag{1.6}$$

assuming initial data

$$u_0 \in C^{2,\alpha}(\overline{\Omega}), \quad \text{for some } \alpha \in (0, 1) \tag{1.7}$$

and

$$\frac{\partial u_0}{\partial \vec{n}} = 0, \quad x \in \partial\Omega. \quad (1.8)$$

In Section 2 we study the global existence of the solutions (u, v) of system (1.1)–(1.8). The main result is enclosed in the following theorem.

Theorem 1.1. *Under assumptions (1.5)–(1.8), for any $T < \infty$, there exists a constant $C(u_0, \chi, p, \Omega)$, independent of T , such that*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

In Section 3 we consider the steady states of the problem (1.1)–(1.5) and we prove the existence of infinitely many solutions in one dimensional bounded domain for $p \in (1, 2)$.

2. Global existence of solutions

To prove Theorem 1.1 we start with the following result.

Notice that, after integration in (1.1), we have that the total mass is preserved in time, i.e.,

$$\int_{\Omega} u = \int_{\Omega} u_0 = |\Omega|M. \quad (2.1)$$

Remark 2.1. Let $\Omega \subset \mathbb{R}^N$, be a bounded and regular domain and $f \in L^1(\Omega)$, such that $\int_{\Omega} f = 0$, then, the problem

$$\begin{cases} -\Delta v = f, & \text{in } \Omega \\ \frac{\partial v}{\partial \vec{n}} = 0, & \text{in } \partial\Omega, \end{cases}$$

has an unique solution

$$v \in W^{1,s}(\Omega), \quad \text{for} \quad \begin{cases} s \in [1, N/(N-1)), & \text{if } N > 1 \\ s = \infty & \text{if } N = 1, \end{cases} \quad (2.2)$$

satisfying

$$\int_{\Omega} v = 0.$$

The proof of (2.2) is given in Chabrowski [15] Theorem 2.8, where the general problem

$$\begin{cases} -\Delta v = \lambda v + f, & \text{in } \Omega \\ \frac{\partial v}{\partial \vec{n}} = 0, & \text{in } \partial\Omega, \end{cases}$$

is studied for $\lambda \in \mathbb{R}$.

Lemma 2.2. Let $p > 1$ such that

$$\begin{cases} p < \infty, & \text{if } N = 1, \\ 2(p - 1) < \frac{N}{N-1}, & \text{if } N \geq 2, \end{cases} \tag{2.3}$$

then, for any $q > 1$ and any $s > 0$ satisfying

$$\begin{cases} s \in (2(p - 1), \infty), & \text{if } N = 1, \\ s \in \left(2(p - 1), \frac{N}{N-1}\right), & \text{if } N \geq 2, \end{cases} \tag{2.4}$$

the following inequality holds:

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{3(q - 1)}{q} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq c_1 q (q - 1) \chi^2 \left[\int_{\Omega} u^{\frac{sq}{s-2(p-1)}} \right]^{\frac{s-2(p-1)}{s}}, \tag{2.5}$$

for some positive constant c_1 .

Proof. We multiply equation (1.1) by u^{q-1} (for $q > 1$) and integrate by parts to obtain

$$\frac{d}{dt} \frac{1}{q} \int_{\Omega} u^q + (q - 1) \int_{\Omega} |\nabla u|^2 u^{q-2} = (q - 1) \chi \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla u \cdot u^{q-1}. \tag{2.6}$$

Thanks to Young’s inequality, it results

$$\begin{aligned} & \frac{d}{dt} \frac{1}{q} \int_{\Omega} u^q + \frac{4(q - 1)}{q^2} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq \frac{2(q - 1)\chi}{q} \int_{\Omega} |\nabla v|^{p-1} |\nabla u^{\frac{q}{2}}| u^{\frac{q}{2}} \\ & \leq \frac{(q - 1)}{q^2} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 + (q - 1) \chi^2 \int_{\Omega} |\nabla v|^{2p-2} u^q \\ & \leq \frac{(q - 1)}{q^2} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 + (q - 1) \chi^2 \left[\int_{\Omega} |\nabla v|^s \right]^{\frac{2(p-1)}{s}} \left[\int_{\Omega} u^{\frac{sq}{s-2(p-1)}} \right]^{\frac{s-2(p-1)}{s}}, \end{aligned}$$

then

$$\frac{d}{dt} \frac{1}{q} \int_{\Omega} u^q + \frac{3(q - 1)}{q^2} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq (q - 1) \chi^2 \left[\int_{\Omega} |\nabla v|^s \right]^{\frac{2(p-1)}{s}} \left[\int_{\Omega} u^{\frac{sq}{s-2(p-1)}} \right]^{\frac{s-2(p-1)}{s}}. \tag{2.7}$$

We notice that, since $u \in L^1(\Omega)$, we have $v \in W^{1,s}(\Omega)$ for any s satisfying (2.4) and, thanks to Lemma 2.2, we have

$$\|\nabla v\|_{L^s(\Omega)}^{2(p-1)} \leq c_1 < \infty. \quad (2.8)$$

Then, equation (2.7) becomes

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{3(q-1)}{q} \int_{\Omega} \left| \nabla u^{\frac{q}{2}} \right|^2 \leq \chi^2 (q-1) q c_1 \left[\int_{\Omega} u^{\frac{sq}{s-2(p-1)}} \right]^{\frac{s-2(p-1)}{s}},$$

the proof is done. \square

Lemma 2.3. *Under assumption*

$$\begin{cases} p \in (1, \infty), & \text{if } N = 1, \\ p \in \left(1, \frac{N}{N-1}\right), & \text{if } N \geq 2, \end{cases} \quad (2.9)$$

and any $s > 0$ satisfying

$$\begin{cases} s \in (2(p-1), \infty), & \text{if } N = 1, \\ s \in \left(2(p-1), \frac{N}{N-1}\right), & \text{if } N \geq 2, \end{cases} \quad (2.10)$$

we have that

$$\int_{\Omega} u^{N+1} \leq c_5, \quad \text{for any } t > 0, \quad (2.11)$$

and c_5 independent of t .

Proof. We first notice that for $N \geq 2$ we have that $p < 2$ and then $p \geq 2(p-1)$ and, in view of (2.9),

$$2(p-1) \leq p < \frac{N}{N-1}$$

which gives a non-empty set of admissible values of s for $N \geq 2$.

To prove the Lemma we follow an Moser–Alikakos iteration method, the result could be also obtained using similar arguments as in Tao and Winkler [32], for readers convenience we detail the proof. We recall the expression (2.5)

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{3(q-1)}{q} \int_{\Omega} \left| \nabla u^{\frac{q}{2}} \right|^2 \leq \chi^2 (q-1) q c_1 \left[\int_{\Omega} u^{\frac{sq}{s-2(p-1)}} \right]^{\frac{s-2(p-1)}{s}}.$$

By Gagliardo–Nirenberg inequality (see Henry [18]), for $1 \leq \gamma \leq \beta \leq \infty, r \geq 1$ and

$$-\frac{N}{\beta} = a\left(1 - \frac{N}{r}\right) - \frac{N}{\gamma}(1 - a),$$

if

$$a \in (0, 1),$$

we have that

$$\|w\|_{L^\beta(\Omega)} \leq C_{GN} \|w\|_{L^\gamma(\Omega)}^{1-a} \|w\|_{W^{1,r}(\Omega)}^a. \tag{2.12}$$

Notice that

$$a = \frac{\frac{1}{\gamma} - \frac{1}{\beta}}{\frac{1}{\gamma} + \frac{1}{N} - \frac{1}{r}}.$$

We take

$$w = u^{\frac{q}{2}}, \quad \gamma = 1 \quad \beta = \frac{2s}{s - 2(p - 1)}, \quad r = 2, \quad \text{and } s = \frac{N}{N - 1 + \epsilon}$$

for ϵ small enough such that

$$s \in \left(2(p - 1), \frac{N}{N - 1}\right), \quad N - \frac{p}{p - 1} + \epsilon < 0. \tag{2.13}$$

Then, we obtain

$$\|u^{\frac{q}{2}}\|_{L^{\frac{2s}{s-2(p-1)}}(\Omega)} \leq C_{GN} \|u^{\frac{q}{2}}\|_{H^1(\Omega)}^a \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^{1-a},$$

for

$$\begin{aligned} a &= \frac{1 - \frac{s-2(p-1)}{2s}}{\frac{1}{2} + \frac{1}{N}} = \frac{2N}{N+2} \left(1 - \frac{s-2(p-1)}{2s}\right) = \frac{N}{N+2} \frac{(s+2(p-1))}{s} \\ &= \frac{N}{N+2} \frac{\left(\frac{N}{N-1+\epsilon} + 2(p-1)\right)}{\frac{N}{N-1+\epsilon}}, \end{aligned}$$

equivalent to

$$\begin{aligned} a &= \frac{N - 1 + \epsilon}{N + 2} \left(\frac{N}{N - 1 + \epsilon} + 2(p - 1)\right) = \frac{N + 2(p - 1)(N - 1 + \epsilon)}{N + 2} \\ &< \frac{N + 2 + 2(p - 1)N - 2p + 2(p - 1)\epsilon}{N + 2} = 1 + 2(p - 1) \frac{N - \frac{p}{p-1} + \epsilon}{N + 2}. \end{aligned}$$

Notice that

$$N < \frac{p}{p-1} \quad \Longleftrightarrow \quad p < \frac{N}{N-1},$$

and in view of (2.13) we have $a < 1$.

We now apply inequality (2.12) to the last term in (2.5) to obtain

$$\left[\int_{\Omega} u^{\frac{sq}{s-2(p-1)}} \right]^{\frac{s-2(p-1)}{s}} \leq C_{GN} \left[\|u^{\frac{q}{2}}\|_{H^1(\Omega)}^a \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^{1-a} \right]^2$$

and

$$C_{GN} \left[\|u^{\frac{q}{2}}\|_{H^1(\Omega)}^a \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^{1-a} \right]^{\frac{s-2(p-1)}{s}} \leq \delta \|u^{\frac{q}{2}}\|_{H^1(\Omega)}^2 + \frac{c_2}{\delta^{\frac{a}{1-a}}} \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^2,$$

for some positive δ small enough. Using the above estimates, (2.5) becomes

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{3(q-1)}{q} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq \chi^2 (q-1) q c_1 \left[\delta \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 + \delta \int_{\Omega} u^q + \frac{c_2}{\delta^{\frac{a}{1-a}}} \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^2 \right],$$

i.e.,

$$\frac{d}{dt} \int_{\Omega} u^q + \left[\frac{3(q-1)}{q} - \delta \chi^2 (q-1) q c_1 \right] \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq \chi^2 (q-1) q c_1 \left[\delta \int_{\Omega} u^q + \frac{c_2}{\delta^{\frac{a}{1-a}}} \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^2 \right]$$

and for $\delta < \frac{2}{\chi^2 c_1 q^2}$ it results

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{(q-1)}{q} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq \chi^2 (q-1) q c_1 \left[\delta \int_{\Omega} u^q + \frac{c_2}{\delta^{\frac{a}{1-a}}} \|u^{\frac{q}{2}}\|_{L^1(\Omega)}^2 \right]. \quad (2.14)$$

Thanks to Poincaré–Wirtinger inequality we get

$$\int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \geq C_{PW} \int_{\Omega} \left(u^{\frac{q}{2}} - \frac{1}{|\Omega|} \int_{\Omega} u^{\frac{q}{2}} \right)^2 = C_{PW} \left(\int_{\Omega} u^q - \frac{1}{|\Omega|} \left| \int_{\Omega} u^{\frac{q}{2}} \right|^2 \right),$$

which implies

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{q-1}{q} \left[\frac{C_{PW}}{2} - \delta q^2 c_1 \chi^2 \right] \int_{\Omega} u^q \leq c_3(\delta, |\Omega|, q) \left| \int_{\Omega} u^{\frac{q}{2}} \right|^2.$$

For $\delta < \min\{\frac{C_{PW}}{4c\chi^2q^2}, \frac{2}{\chi^2c_1q^2}\}$ the last inequality is reduced to

$$\frac{d}{dt} \int_{\Omega} u^q + \frac{q-1}{q} \frac{C_{PW}}{4} \int_{\Omega} u^q \leq c_3(\delta, |\Omega|, q) \left| \int_{\Omega} u^{\frac{q}{2}} \right|^2 \tag{2.15}$$

and by the Maximum Principle, we get

$$\sup_{t>0} \|u\|_{L^q(\Omega)}^q \leq \frac{4qc_3(\delta, |\Omega|, q)}{(q-1)C_{PW}} \sup_{t>0} \|u\|_{L^{\frac{q}{2}}(\Omega)}^q. \tag{2.16}$$

Following Moser–Alikakos iteration (see [1]), we define

$$x_i := \sup_{t>0} \int_{\Omega} u^{2^i}, \quad i \in \mathbb{N}.$$

Thanks to (2.1) it results $x_0 = |\Omega|M < \infty$ and (2.16) implies that

$$x_i \leq c_4(\delta, i)x_{i-1}^2.$$

Notice that x_i is finite for $i < \infty$, in particular, there exists i_0 large enough, such that $2^{i_0} > N + 1$ and therefore

$$\int_{\Omega} u^{N+1} \leq c_5 < \infty \quad \text{for any } t > 0,$$

and the proof ends. \square

Lemma 2.4. *There exists a positive constant c_6 , independent of t , such that the following bound holds*

$$\|\nabla v\|_{L^\infty(\Omega)} < c_6.$$

Proof. In view of (2.11) we claim that $v \in W^{2,N+1}(\Omega) \subset W^{1,\infty}(\Omega)$, thanks to Chabrowski [15] Theorem 2.8, we have the result. \square

Lemma 2.5. *There exists a positive constant c_∞ , independent of t , such that*

$$\|u\|_{L^\infty(\Omega)} \leq c_\infty.$$

Proof. From (2.6) (for $q \geq 2$) and Lemma 2.4 we have

$$\frac{d}{dt} \frac{1}{q} \int_{\Omega} u^q + (q-1) \int_{\Omega} |\nabla u|^2 u^{q-2} = (q-1)c_7 \int_{\Omega} |\nabla u| u^{q-1}.$$

Since

$$c_7 \int_{\Omega} |\nabla u| u^{q-1} \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 u^{q-2} + \frac{c_7^2}{2} \int_{\Omega} u^q,$$

we get

$$\frac{d}{dt} \frac{1}{q} \int_{\Omega} u^q + \frac{(q-1)}{2} \int_{\Omega} |\nabla u|^2 u^{q-2} = (q-1) \frac{c_7^2}{2} \int_{\Omega} u^q. \quad (2.17)$$

In the Gagliardo–Nirenberg’s inequality

$$\|w\|_{L^2(\Omega)} \leq C_{GN1} \|\nabla w\|_{L^2(\Omega)}^a \|w\|_{L^1(\Omega)}^{1-a} + C_{GN2} \|w\|_{L^1(\Omega)}, \quad \text{for } \frac{1}{2} = a \left(\frac{1}{2} - \frac{1}{N} \right) + 1 - a,$$

taking

$$w = u^{\frac{q}{2}}, \quad \text{for } a = \frac{N}{N+2} < 1,$$

we obtain

$$\left[\int_{\Omega} u^q \right]^{\frac{1}{2}} \leq C_{GN1} \left[\int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \right]^{\frac{a}{2}} \left[\int_{\Omega} u^{\frac{q}{2}} \right]^{(1-a)} + C_{GN2} \int_{\Omega} u^{\frac{q}{2}},$$

which is equivalent to

$$\int_{\Omega} u^q \leq 2C_{GN1} \left[\int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \right]^a \left[\int_{\Omega} u^{\frac{q}{2}} \right]^{2(1-a)} + 2C_{GN2} \left[\int_{\Omega} u^{\frac{q}{2}} \right]^2.$$

Replacing the last expression in (2.17) we have

$$\frac{d}{dt} \frac{1}{q} \int_{\Omega} u^q + \frac{2(q-1)}{q^2} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \leq (q-1)c_8 \left(\left[\int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \right]^a \left[\int_{\Omega} u^{\frac{q}{2}} \right]^{2(1-a)} + \left[\int_{\Omega} u^{\frac{q}{2}} \right]^2 \right) \quad (2.18)$$

and applying Young’s inequality to the first term of the right side,

$$\left[\int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 \right]^a \left[\int_{\Omega} u^{\frac{q}{2}} \right]^{2(1-a)} \leq \frac{1}{q^2 c_8} \int_{\Omega} |\nabla u^{\frac{q}{2}}|^2 + [q^2 c_8 a]^{\frac{a}{(1-a)}} (1-a) \left[\int_{\Omega} u^{\frac{q}{2}} \right]^2,$$

inequality (2.18) is reduced to

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} u^q + \frac{(q-1)}{q^2} \int_{\Omega} \left| \nabla u^{\frac{q}{2}} \right|^2 \leq (q-1) q^{\frac{2a}{(1-a)}} c_9 \left[\int_{\Omega} u^{\frac{q}{2}} \right]^2,$$

i.e.,

$$\frac{q}{(q-1)} \frac{d}{dt} \int_{\Omega} u^q + \int_{\Omega} \left| \nabla u^{\frac{q}{2}} \right|^2 \leq q^{\frac{2}{(1-a)}} c_9 \left[\int_{\Omega} u^{\frac{q}{2}} \right]^2$$

Thanks to Poincaré–Wirtinger’s inequality we have

$$\int_{\Omega} \left| \nabla u^{\frac{q}{2}} \right|^2 \geq C_{PW} \int_{\Omega} \left(u^{\frac{q}{2}} - \frac{1}{|\Omega|} \int_{\Omega} u^{\frac{q}{2}} \right)^2 = C_{PW} \left(\int_{\Omega} u^q - \frac{1}{|\Omega|} \left| \int_{\Omega} u^{\frac{q}{2}} \right|^2 \right),$$

which implies

$$\frac{q}{q-1} \frac{d}{dt} \int_{\Omega} u^q + C_{PW} \int_{\Omega} u^q \leq q^{2+\frac{2a}{(1-a)}} c_{10} \left| \int_{\Omega} u^{\frac{q}{2}} \right|^2.$$

By the Maximum Principle we get

$$\sup_{t>0} \|u\|_{L^q(\Omega)}^q \leq \frac{q^{\frac{2}{(1-a)}} c_{10}}{C_{PW}} \sup_{t>0} \|u\|_{L^{\frac{q}{2}}(\Omega)}^q. \tag{2.19}$$

As in Lemma 2.3 we apply the following Moser–Alikakos iteration (see [1]), defining

$$x_i := \sup_{t>0} \int_{\Omega} u^{2^i}, \quad i \in \mathbb{N}.$$

Thanks to (2.1) we have that $x_0 = |\Omega|M < \infty$ and (2.16) implies that

$$x_i \leq 2^{\frac{2i}{(1-a)}} c_{11} x_{i-1}^2,$$

then

$$x_i^{2^{-i}} \leq 2^{\frac{i2^{1-i}}{(1-a)}} c_{11}^{2^{-i}} x_{i-1}^{2^{1-i}},$$

and

$$\|u\|_{L^{2^i}(\Omega)} \leq \prod_{j=0 \dots i} \frac{j2^{1-j}}{2(1-a) c_{11}^{2^{-j}}} = 2^{\sum_{j=0 \dots i} \frac{j2^{1-j}}{(1-a)}} \sum_{c_{11}^{j=0 \dots i}} 2^{-j}.$$

Since

$$\sum_{j=0\dots i} \frac{j2^{1-j}}{(1-a)} = \frac{2}{(1-a)} \sum_{j=0\dots i} j2^{-j} = \frac{2}{(1-a)} (2 - 2^{-i} - 2^{-i-1}) \leq \frac{4}{(1-a)}$$

and

$$\sum_{j=0\dots i} 2^{-j} = 2 - 2^{-i} < 2,$$

we obtain

$$\|u\|_{L^{2i}(\Omega)} \leq 2^{\frac{2}{(1-a)}} c_{11}^2 := c_{\infty}.$$

Constant c_{11} is independent of i , taking limits when $i \rightarrow \infty$, the proof ends. \square

End of the proof of Theorem 1.1.

The proof is a consequence of the previous lemma.

Remark 2.6. We notice that, if equation (1.2) is replaced by

$$-\Delta v + v = u, \quad x \in \Omega, \quad t > 0, \quad (2.20)$$

and Remark 2.1 is replaced by Lemma 23 in Brezis–Strauss [12], we also obtain the boundedness in $L^{\infty}(\Omega)$ of u and v and the global existence of solutions.

3. Stationary states in 1 D

The steady states of the problem (1.1)–(1.5) for $\Omega = (0, 1)$, are given by

$$\begin{cases} -u_{xx} = -(\chi u |v_x|^{p-2} v_x)_x, & x \in (0, 1), \\ -v_{xx} = u - M, & x \in (0, 1), \\ u_x(0) = u_x(1) = v_x(0) = v_x(1) = 0. \end{cases} \quad (3.1)$$

The aim of this section is to prove the following theorem:

Theorem 3.1. *Let $p \in (1, 2)$, then, for any $M > 0$ and $\chi > 0$, there exist infinitely many solutions to (3.1).*

In order to prove the theorem, we proceed into several steps.

We consider the following variables

$$w := \ln(u) - \ln(M), \quad \beta := v_x \quad (3.2)$$

then, (3.1) becomes

$$\begin{cases} w_x = \chi |\beta|^{p-2} \beta, & x \in (0, 1), \\ \beta_x = M(1 - e^w), & x \in (0, 1), \\ \beta(0) = \beta(1) = 0. \end{cases} \tag{3.3}$$

We have that (3.3) is a Hamiltonian system, i.e., there exists $H(w, \beta)$ such that

$$\begin{cases} \chi |\beta|^{p-2} \beta = \frac{\partial H(w, \beta)}{\partial \beta}, & x \in (0, 1) \\ M(1 - e^w) = -\frac{\partial H(w, \beta)}{\partial w}, & x \in (0, 1). \end{cases} \tag{3.4}$$

After integration, we obtain

$$H(w, \beta) = M(e^w - w - 1) + \chi \frac{|\beta|^p}{p}. \tag{3.5}$$

The solutions of the system are found along the contours of H , so that, to sketch the phase diagram of this system, it is enough to study and draw the level sets of the Hamiltonian function H . We notice that there exists an energy in the system (3.3) which is preserved along the solutions, i.e.,

$$\frac{M}{\chi}(e^w - w - 1) + \frac{1}{p}|\beta|^p = k, \tag{3.6}$$

where k is a non negative constant. Notice that for $k = 0$ we have the trivial solution $w = \beta = 0$.

In the following lemma we prove the conservation of the energy.

Lemma 3.2. *Let (w, β) be a solution to (3.3) and the hamiltonian function H as in (3.5), i.e.,*

$$H(w, \beta) = M(e^w - w - 1) + \frac{\chi}{p}|\beta|^p$$

then,

$$\frac{dH}{dx} = 0.$$

Proof. Recalling the Hamilton function H is conserved in any solution of the system, i.e.,

$$\frac{dH(w(x), \beta(x))}{dx} = \frac{\partial H}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial H}{\partial \beta} \frac{\partial \beta}{\partial x} = \frac{\partial H}{\partial w} \frac{\partial H}{\partial \beta} - \frac{\partial H}{\partial \beta} \frac{\partial H}{\partial w} = 0,$$

and we have the proof. \square

Notice that the level sets of H are bounded curves corresponding to a periodic solution (w, β) of period $T(k)$, for k defined in (3.6).

End of the proof of Theorem 3.1.

We write $|\beta|$ in terms of $|w_x|$, then

$$|\beta|^p = \chi \frac{p}{1-p} |w_x|^{\frac{p}{p-1}}$$

and

$$\frac{M}{\chi}(e^w - w - 1) + \frac{\chi^{p1-p}}{p}|w_x|^{\frac{p}{p-1}} = k, \quad (3.7)$$

and $|w_x|$ satisfies

$$|w_x| = [kp\chi^{\frac{p}{p-1}} + Mp\chi^{\frac{1}{p-1}}(1 + w - e^w)]^{\frac{p-1}{p}}. \quad (3.8)$$

We denote by r_0 and $-r_1$ the values of w for $\beta = 0$ for a given k , then r_1 and r_0 satisfy

$$k = \frac{M}{\chi}(e^{r_0} - r_0 - 1) = \frac{M}{\chi}(e^{-r_1} + r_1 - 1).$$

After integration in (3.8) we get

$$\int_{-r_1}^{r_0} [kp\chi^{\frac{p}{p-1}} + Mp\chi^{\frac{1}{p-1}}(1 + w - e^w)]^{\frac{1-p}{p}} dw = [Mp\chi^{\frac{1}{p-1}}]^{\frac{p-1}{p}} \frac{T}{2}, \quad (3.9)$$

and

$$\begin{aligned} \int_{-r_1}^{r_0} [kp\chi^{\frac{p}{p-1}} + Mp\chi^{\frac{1}{p-1}}(1 + w - e^w)]^{\frac{p-1}{p}} dw &= \left| \frac{M}{\chi} \right|^{-\frac{p-1}{p}} \int_{-r_1}^0 [e^{-r_1} + r_1 - e^w + w]^{-\frac{p-1}{p}} dw \\ &\quad + \left| \frac{M}{\chi} \right|^{-\frac{p-1}{p}} \int_0^{r_0} [e^{r_0} - r_0 - e^w + w]^{-\frac{p-1}{p}} dw \end{aligned}$$

To estimate the righthand side integrals, we apply Hölder inequality, so, for any $\epsilon > 0$ verifying

$$\epsilon < \frac{2-p}{p-1} \quad (3.10)$$

we have:

$$\begin{aligned} \int_0^{r_0} [e^{r_0} - r_0 - e^w + w]^{-\frac{p-1}{p}} dw &\leq \int_0^{r_0} (e^w - 1) [e^{r_0} - r_0 - e^w + w]^{-\frac{(p-1)(2+\epsilon)}{p}} dw \\ &\quad + \int_0^{r_0} (e^w - 1)^{-\frac{1}{1+\epsilon}}. \end{aligned}$$

After integration, for $p < 2$, we get

$$\int_0^{r_0} [e^{r_0} - r_0 - e^w + w]^{-\frac{(p-1)(2+\epsilon)}{p}} dw = \frac{p}{(2+\epsilon) - p(1+\epsilon)} [e^{r_0} - r_0 - 1]^{-\frac{(2+\epsilon)-p(1+\epsilon)}{p}}$$

and

$$\int_0^{r_0} (e^w - 1)^{-\frac{1}{1+\epsilon}} \leq \int_0^{r_0} w^{-\frac{1}{1+\epsilon}} = \frac{1+\epsilon}{\epsilon} r_0^{\frac{\epsilon}{1+\epsilon}}.$$

In the same way we obtain

$$\begin{aligned} \int_{-r_1}^0 [e^{-r_1} + r_1 - e^w + w]^{-\frac{p-1}{p}} dw &\leq \int_{-r_1}^0 (1 - e^w) [e^{-r_1} + r_1 - e^w + w]^{-\frac{(p-1)(2+\epsilon)}{p}} dw \\ &+ \int_{-r_1}^0 (1 - e^w)^{-\frac{1}{1+\epsilon}} dw. \end{aligned}$$

Computing the above integrals it follows

$$\begin{aligned} &\int_{-r_1}^0 (1 - e^w) [e^{-r_1} + r_1 - e^w + w]^{-\frac{(p-1)(2+\epsilon)}{p}} dw \\ &= \frac{p}{(2+\epsilon) - p(1+\epsilon)} [e^{-r_1} + r_1 - 1]^{1-\frac{(p-1)(2+\epsilon)}{p}} \end{aligned}$$

and

$$\int_{-r_1}^0 (1 - e^w)^{-\frac{1}{1+\epsilon}} dw = \int_0^{r_1} (1 - e^{-w})^{-\frac{1}{1+\epsilon}} dw.$$

In view of

$$1 - e^{-w} \geq \frac{1 - e^{-r_1}}{r_1} x, \quad x \in (0, r_1)$$

it results

$$\int_0^{r_1} (1 - e^{-w})^{-\frac{1}{1+\epsilon}} dw \leq (1 - e^{-r_1})^{-\frac{1}{1+\epsilon}} r_1^{\frac{1}{1+\epsilon}} \int_0^{r_1} \frac{1}{x^{\frac{1}{1+\epsilon}}} dx = \frac{\epsilon}{1+\epsilon} (1 - e^{-r_1})^{-\frac{1}{1+\epsilon}} r_1.$$

Therefore

$$\begin{aligned} \frac{T}{2} \leq & \left[\frac{M}{\chi} \right]^{-\frac{p-1}{p}} \left[\frac{p}{(2+\epsilon) - p(1+\epsilon)} [e^{r_0} - r_0 - 1]^{\frac{(2+\epsilon)-p(1+\epsilon)}{p}} + \frac{1+\epsilon}{\epsilon} r_0^{\frac{\epsilon}{1+\epsilon}} \right. \\ & \left. + \frac{p}{(2+\epsilon) - p(1+\epsilon)} [e^{-r_1} + r_1 - 1]^{\frac{(2+\epsilon)-p(1+\epsilon)}{p}} + \frac{\epsilon}{1+\epsilon} (1 - e^{-r_1})^{-\frac{1}{1+\epsilon}} r_1 \right] \end{aligned}$$

which implies

$$\lim_{k \rightarrow 0} T = \lim_{r_1, r_0 \rightarrow 0} T = 0, \quad \text{for } p < 2.$$

To determine $\lim_{k \rightarrow \infty} T$ we apply the definition of T

$$T = 2[Mp\chi^{\frac{1}{p-1}}]^{\frac{1-p}{p}} \int_{-r_1}^{r_0} \left[\frac{\chi}{M}k - (1+w-e^w) \right]^{\frac{1-p}{p}} dw.$$

The following bound holds

$$\begin{aligned} T & \geq C \int_{\frac{r_1}{2}}^{r_1} [r_1 + e^{-r_1} - w - e^{-w}]^{\frac{1-p}{p}} dw \geq \frac{r_1}{2} \min_{w \in (r_1/2, r_1)} [r_1 + e^{-r_1} - w - e^{-w}]^{\frac{1-p}{p}} \\ & = \frac{r_1}{2} \left[\frac{r_1}{2} + e^{-r_1} - e^{-r_1/2} \right]^{\frac{1-p}{p}}, \end{aligned}$$

and for $r_1 > 2$ we have that

$$\frac{r_1}{2} + e^{-r_1} - e^{-r_1/2} \leq \frac{r_1}{2} + 1 \leq r_1.$$

Applying it in the previous inequality that verifies T , we get

$$T \geq cr_1^{\frac{1}{p}}$$

and then

$$\lim_{k \rightarrow \infty} T = \lim_{r_1 \rightarrow \infty} T = \infty, \quad \text{for } p < 2.$$

By the continuity of T with respect to k , there exist $k_n > 0$ such that T_n , the period of the corresponding solution (w_n, β_n) to the energy constant k_n , satisfies

$$T_n = \frac{1}{n}$$

and therefore

$$w'_n(0) = w'_n(1) = 0.$$

We recover the solution of the original problem (u_n, v_n) defined by

$$u_n(x) = Me^{w_n(x)}, \quad v_n(x) = \int_0^x \beta_n(s)ds - \int_0^1 \beta_n(s)ds$$

and the proof ends. \square

Remark 3.3. We notice that, the steady states for the 1-dimensional problem (1.2)–(1.4), defined by (3.1), are equivalent to the solutions of the problem

$$\begin{cases} -u_{xx} = -\chi(uw_x)_x, & x \in \Omega, \\ -(|w_x|^{\frac{2-p}{p-1}}w_x)_x = u - M, & x \in \Omega, \\ w_x(0) = w_x(1) = 0. \end{cases}$$

Since $u = Me^{\chi w}$, we have

$$\begin{cases} -(|w_x|^{\frac{2-p}{p-1}}w_x)_x = M(e^{\chi w} - 1), & x \in \Omega, \\ w_x(0) = w_x(1) = 0, \end{cases}$$

for w satisfying

$$\int_{\Omega} e^{\chi w} = 1.$$

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