

## ON A PARABOLIC-ODE SYSTEM OF CHEMOTAXIS

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**ABSTRACT.** In this article we consider a coupled system of differential equations to describe the evolution of a biological species. The system consists of two equations, a second order parabolic PDE of nonlinear type coupled to an ODE. The system contains chemotactic terms with constant chemotaxis coefficient describing the evolution of a biological species “ $u$ ” which moves towards a higher concentration of a chemical species “ $v$ ” in a bounded domain of  $\mathbb{R}^n$ . The chemical “ $v$ ” is assumed to be a non-diffusive substance or with neglectable diffusion properties, satisfying the equation

$$v_t = h(u, v).$$

We obtain results concerning the bifurcation of constant steady states under the assumption

$$h_v + \chi u h_u > 0$$

with growth terms  $g$ . The Parabolic-ODE problem is also considered for the case  $h_v + \chi u h_u = 0$  without growth terms, i.e.  $g \equiv 0$ . Global existence of solutions is obtained for a range of initial data.

**1. Introduction.** Haptotaxis is the directional motility of living organisms towards a gradient of a chemical concentration by adhesion, in contrast to chemotaxis, where the chemoattractant develops in a soluble environment. The process has been extensively studied from a biological point of view from the invention of microscope in the XIX century. In the last decades, several mathematical models have been presented to describe the phenomenon, after the pioneering works of Patlak [36] and Keller and Segel [14] in chemotaxis (see also the review articles Horstmann [12], [13] or Bellomo et al [5]), haptotaxis models have been presented in the last decades, see for instance Othmer and Stevens [37], Levine and Sleeman [22], Stevens [40] and Anderson and Chaplain [1], among others.

The original model in [14] describes the evolution of a biological species, denoted by “ $u$ ” and the distribution of chemical concentration “ $v$ ” in a coupled system of

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two second order parabolic equations. The system is presented as follows

$$\begin{cases} u_t = d_u \Delta u - \operatorname{div}(u\chi(v)\nabla v) + g(u, v), \\ v_t = d_v \Delta v + h(u, v), \end{cases} \quad (1)$$

with appropriate boundary conditions and initial data. In (1), diffusion is considered for the chemical, nevertheless, several biological processes involving directional movement towards a higher concentration of chemical agents present non-diffusive chemical substance. In Anderson and Chaplain [1], a system of 3 equations is proposed to describe angiogenesis in tumor growth. The system in [1] considers the concentration of endothelial cells, whose movement towards a higher concentration of Tumor Angiogenesis Factor (TAF) depends on the distribution of fibronectin in the extracellular matrix. The fibronectin is a molecule set in the extracellular matrix which doesn't present diffusion. In [1], the concentration of fibronectin is described in terms of an ODE, see also Sleeman and Levine [38], Kubo and Suzuki [19] and Kubo, Hoshino and Kimura [18] for Parabolic-ODE systems of angiogenesis with chemotactic terms.

Haptotaxis is also presented in other biological processes, as morphogenesis, the formation of organs and shapes in the embryo of animals. Starting with the pioneering work of Turing [47], several models of PDEs have been introduced to describe the process in developing biology, see for instance Malogrosz [26] and [27], Tello [45], Muñoz and Tello [31], Krzyzanowski, P. Laurençot and P. Wrzosek [17]. In Merking and Sleeman [28], Merking, Needham and Sleeman [29] and Bollenbach et al [4], models of morphogenesis are described in terms of PDEs with chemotactic terms, see also Stinner, Tello and Winkler [41] for the mathematical results concerning global existence and linear stability of the model proposed in [4].

The mathematical model of haptotaxis doesn't present diffusion term in the equation, as the Keller-Segel model does. The system is as follows

$$\begin{cases} u_t - \Delta u = -\operatorname{div}(\chi u \nabla v) + g(u, v), & \text{in } \Omega \times (0, T), \\ v_t = h(u, v), & \text{in } \Omega \times (0, T) \\ \frac{\partial u}{\partial \vec{n}} = 0, & \text{in } \partial\Omega, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & \text{in } \Omega \end{cases} \quad (2)$$

where the symbol  $\partial/\partial\vec{n}$  denotes the derivative with respect to the outer normal of  $\partial\Omega$ . To complete the problem, the initial data satisfy

$$u_0, v_0 \in C^{2+\alpha}(\Omega), \quad \frac{\partial v_0}{\partial \vec{n}} = 0, \quad x \in \partial\Omega \quad (3)$$

for some  $\alpha \in (0, 1)$ . In the above equations we have considered that  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary  $\partial\Omega$ .

System (2) is studied in Friedman and Tello [10], under assumptions

$$h_u u \chi + h_v < 0, \quad (4)$$

without growth terms (i.e.  $g \equiv 0$ ). In [10], the global existence and the asymptotic behavior is obtained for a general case of non-constant coefficients  $\chi$ .

The model proposed in Levine, Sleeman and Nilsen-Hamilton [24] is studied in Fontelos, Friedman and Hu [9] for an one-dimensional domain.

The system for  $g \equiv 0$  has been also studied in Suzuki [43], [44] and Sleeman and Levine [39], for different type of functions  $h$  which appears in the context of a tumor

growth. In [39] the stability of non-constant solutions is also studied for the case

$$v_t - \epsilon \Delta v = h(u, v).$$

In Kubo and Tello [20] the system with competitive terms is studied when

$$g(u, v) = \mu_1 u(1 - u - a_1 v), \quad h(u, v) = \mu_2 u(1 - a_2 u - v),$$

for weak competition, i.e.,  $a_i \in (0, 1)$ , for  $i = 1, 2$ . Under suitable assumptions for the chemotactic coefficients, the global existence of solutions and the stability of the constant steady states are obtained by using an energy method.

Systems with several biological species have been also considered from a mathematical point of view, see Lauffenburger [21], Fasano et al [8], Conca and Espejo [6], Black [2], Black et al [3], Hirata, Kurima, Mizukami, Yokota [11], Tello and Winkler [46], Stinner, Tello and Winkler [42], Negreanu and Tello [34], [32] Wang and Wu [48], among others. In Negreanu and Tello [35], the following haptotaxis system is considered

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1 - u), & x \in \Omega, t > 0, \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1 - v), & x \in \Omega, t > 0, \\ w_t = h(u, v, w), & x \in \Omega, t > 0, \end{cases}$$

where  $h$  satisfies

$$\frac{\partial h}{\partial u} \geq \epsilon_u > 0 \quad \text{and} \quad \frac{\partial h}{\partial v} \geq \epsilon_v > 0, \quad \frac{\partial h}{\partial w} < 0$$

and

$$0 < k_{0i} \leq \chi_i(w) e^{\int_w^w \chi_i(s) ds}, \quad (i=1,2).$$

In [35], the authors obtain the global existence and the asymptotic behavior of the solutions for a range of parameters and initial data. See also [30] and [25].

In this article we consider the situation, where assumption (4) is not satisfied. Two cases are studied

Case I.  $h_u u \chi + h_v > 0$ .

Case II.  $h_u u \chi + h_v = 0$ .

Case I is considered with growth terms  $g$  under assumptions:

There exists  $(u^*, v^*)$  strictly positive homogeneous steady state of the system (2), i.e.,

$$g(u^*, v^*) = 0, \quad h(u^*, v^*) = 0 \tag{5}$$

and there exists an open set  $A \subset \mathbb{R}^2$ , containing the steady state  $(u^*, v^*)$ , such that

$$g, h \in C^2(A), \tag{6}$$

$$h_u > 0, \quad h_v < 0 \quad \text{in } A, \tag{7}$$

$$\frac{\partial h}{\partial v} + \chi u \frac{\partial h}{\partial u} > 0 \quad \text{in } A, \tag{8}$$

$$g_u h_v - g_v h_u > 0 \quad \text{in } A, \tag{9}$$

are satisfied for  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$  for a given  $\chi_n$ .

In Section 2 we consider the stationary states and study the bifurcation of solutions to prove the existence of nontrivial steady states for a range of parameters, the proof is a consequence of a Bifurcation Theorem for Fredholm Operators (see for instance [16], Theorem II 4.4).

Notice that the linear stability of the dynamical system

$$\begin{cases} \frac{du}{dt} = g(u, v), & \text{in } (0, T), \\ \frac{dv}{dt} = h(u, v), & \text{in } (0, T) \end{cases} \quad (10)$$

is defined by the eigenvalues of the matrix

$$\begin{pmatrix} g_u(u^*, v^*) & g_v(u^*, v^*) \\ h_u(u^*, v^*) & h_v(u^*, v^*) \end{pmatrix} \quad (11)$$

whose determinant is given by

$$Det := g_u(u^*, v^*)h_v(u^*, v^*) - g_v(u^*, v^*)h_u(u^*, v^*).$$

The assumption  $h_u u \chi + h_v < 0$  for a regular function  $h$  guarantees the global existence of solutions provided the initial data are regular enough (see Friedman-Tello [10]), nevertheless, if  $h$  satisfies  $h_u u \chi + h_v > 0$ , the global existence of solutions is unclear. Local existence of solutions can be obtained as in [10] by using a fixed point argument. Uniqueness of solutions is a consequence of the regularity of  $h$  and  $g$ .

In Section 3, the parabolic-ode problem is studied for Case II under the assumptions

$$\frac{\partial h}{\partial v} + \chi u \frac{\partial h}{\partial u} = 0, \quad (12)$$

which gives us an expression of  $h$  in the form  $h(u, v) = \tilde{h}(ue^{-\chi v})$ , where  $\tilde{h}$  satisfies

$$\tilde{h} \in C^2(\mathbb{R}), \quad (13)$$

$$\tilde{h}' > 0. \quad (14)$$

There exists a nonnegative homogeneous steady state  $w^* = u^* e^{-\chi v^*}$ , satisfying

$$h(w^*) = 0. \quad (15)$$

The results achieved give global existence of solutions under the assumption

$$-h - u \chi e^{-\chi v} h' < 0, \quad \text{if } |ue^{-\chi v} - w^*| \leq \epsilon$$

for some positive  $\epsilon$ . The proof of the global existence and the asymptotic stability of the solutions for case II is based in the rectangle method. We notice that the method is not profitable in case I.

**2. Steady states: Bifurcation from constant steady states.** The steady states of the system satisfy the equation

$$\begin{cases} -\Delta u = -\text{div}(\chi u \nabla v) + g(u, v), & \text{in } \Omega, \\ h(u, v) = 0, & \text{in } \Omega. \end{cases} \quad (16)$$

By assumption (8) and Implicit Function Theorem we may write  $v$  as a function of  $u$ , i.e.,

$$v = \phi(u). \quad (17)$$

Since  $h(u, \phi(u)) = 0$ , derivating respect to  $u$ , we get

$$\phi'(u) = -\frac{h_u}{h_v}.$$

We replace in the equation to obtain

$$-\text{div}[(1 - \chi u \phi'(u)) \nabla u] = g(u, \phi(u)). \quad (18)$$

Recalling that  $\phi'(u) = -h_u/h_v$  we have

$$- \operatorname{div} \frac{1}{h_v} [(h_v + \chi u h_u(u)) \nabla u] = g(u, \phi(u)). \tag{19}$$

We denote by  $\psi$  the solution to the equation

$$\psi'(u) = \frac{1}{h_v} (h_v + \chi u h_u(u)) = 1 - \chi u \phi'(u).$$

Since  $\psi' \neq 0$  and  $\operatorname{sign}(\psi') = \operatorname{sign}(h_v)$  we write the equation in the following way

$$\begin{cases} -\Delta \psi = F(\psi, \chi), & x \in \Omega, \\ \frac{\partial \psi}{\partial \vec{n}} = 0 & x \in \partial \Omega, \end{cases} \tag{20}$$

where  $F(\psi(u, \chi), \chi) = g(u, \phi(u))$ .

Problem (20) can be expressed as follows

$$G(\psi, \chi) := \Delta \psi + F(\psi, \chi) = 0$$

with  $\psi \in X$ , for

$$X := \left\{ w \in W^{2,p}(\Omega) : \frac{\partial w}{\partial \vec{n}} = 0, \quad \text{for some } p < n \right\}.$$

Let  $\Sigma$  be the set of eigenvalues of  $-\Delta$  with homogeneous Neumann boundary conditions defined on  $X$ , i.e.,

$$\Sigma := \bigcup_{n=0}^{\infty} \{\lambda_n\}$$

and let  $u^*, v^*$  be defined in (5) as the constant steady states of the system. We denote by  $\psi^*$  the corresponding steady states of problem (20) i.e.  $\psi^* = \psi(u^*)$ . In the same way we define  $\phi^*$  by

$$\phi^* = \phi(u^*).$$

**Theorem 2.1.** *Let  $\chi_n$  defined by*

$$\chi_n = \frac{h_v(u^*, v^*)g_u(u^*, v^*) - g_v(u^*, v^*)h_u(u^*, v^*) - h_v(u^*, v^*)\lambda_n}{\lambda_n u^* h_u(u^*, v^*)},$$

and equivalently

$$\frac{h_v(u^*, v^*)g_u(u^*, v^*) - g_v(u^*, v^*)h_u(u^*, v^*)}{h_v(u^*, v^*) + \chi_0 u^* h_u(u^*, v^*)} = \lambda_n,$$

where  $\lambda_n \in \Sigma$  with odd multiplicity. Then, under assumption (6)-(7), there exists a continuous component of solution  $S := (\psi(\chi), \chi)$  to (20) in  $X \times (\chi_n - \epsilon, \chi_n + \epsilon)$  such that  $(\psi^*, \chi_n) \in S$  and  $(\psi(\chi), \chi)$  is a nontrivial solution of (16) for  $\chi \neq \chi_n$ .

*Proof.* We notice that

$$\frac{\partial G}{\partial \psi} \Big|_{\psi=\psi^*} \xi = \Delta \xi + \frac{\partial F}{\partial \psi} \Big|_{\psi=\psi^*} \xi.$$

To compute  $\frac{\partial F}{\partial \psi}$  we consider the following derivative

$$\frac{dF}{du} = \frac{\partial F}{\partial \psi} \frac{\partial \psi}{\partial u} = \frac{\partial F}{\partial \psi} \left( \frac{1}{h_v} (h_v + \chi u h_u(u)) \right).$$

Since  $F(\psi(u), \chi) = g(u, \phi(u))$  and

$$\left. \frac{d}{du} g(u, \phi(u)) \right|_{u=u^*} = g_u + g_v \phi' = \frac{1}{h_v} (h_v g_u - g_v h_u) < 0$$

we get that

$$\frac{\partial F}{\partial \psi} = \frac{h_v g_u - g_v h_u}{h_v + \chi u h_u}. \tag{21}$$

Then

- $(\psi^*, \chi)$  for  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$  is a trivial solution of (20).
- $G : X \times (\chi_n - \epsilon, \chi_n + \epsilon) \times X \rightarrow L^p(\Omega)$  is a continuous function.

- $$\frac{\partial G}{\partial \psi} \in C(X \times (\chi_n - \epsilon, \chi_n + \epsilon) : L(X, L^p(\Omega)))$$

- $$\left. \frac{\partial G}{\partial \psi} \right|_{\psi=\psi^*}$$

satisfies

- is closed for each  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$ ;
- is a Fredholm operator of index zero for each  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$ , i.e.

$$\dim \text{Ker} \left( \left. \frac{\partial G}{\partial \psi} \right|_{\psi=\psi^*} \right) - \text{codim} R \left( \left. \frac{\partial G}{\partial \psi} \right|_{\psi=\psi^*} \right) = 0;$$

- 0 is an isolated eigenvalue.

- $$\left. \frac{\partial G}{\partial \psi} \right|_{\psi=\psi^*}$$

has an odd crossing number at  $\chi = \chi_n$ .

In view of Theorem II.4.4. [16],  $(\psi^*, \chi_n)$  is a bifurcation point and there exists at least a continuous set of nontrivial solutions  $S = (\psi(\chi), \chi)$ , and the proof ends.  $\square$

**Remark 1.** We notice that the case

$$h_v + \chi u h_u < 0 \tag{22}$$

has been studied in Friedman-Tello [10]. (22) implies that

$$\frac{\partial F}{\partial \psi} = \frac{h_v g_u - g_v h_u}{h_v + \chi u h_u(u)} < 0$$

provided (9) is satisfied. Therefore we have uniqueness of solutions and bifurcation may not occur.

**2.1. Examples.** In order to apply Theorem 2.1, odd multiplicity of the eigenvalues of “ $-\Delta$ ” is required. We take  $\Omega = (0, L)$ , for  $L > 0$  which guarantees the odd multiplicity of the eigenvalues and the assumption is satisfied.

Example 1. In [23] the authors studied the following system

$$\begin{cases} u_t - \Delta u = -\text{div}(\chi u \nabla v) + g(u, v), & \text{in } \Omega \times (0, T), \\ v_t = h(u, v), & \text{in } \Omega \times (0, T) \\ \frac{\partial u}{\partial \bar{n}} = 0, & \text{in } \partial\Omega, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & \text{in } \Omega \end{cases}$$

for  $g = 0$  and  $h$  given by

$$h = \frac{k_1 uv}{k_2 + v} - av$$

for positive parameters  $k_1, k_2$  and  $a$ . The system describes the evolution of Dictyostelium discoideum density “ $u$ ” and cAMP (extracellular cyclic Adenosine Monophosphate concentration) produced by the cells. The term is a version of the Michaelis-Menten rate law (see also Edelstein-Keshet [7] p. 276). We include a logistic growth term in the form

$$g(u, v) = \mu u(1 - u),$$

for a positive  $\mu$ . Under assumption

$$k_1 - k_2 a > 0, \quad \chi_0 > \frac{a}{k_1 k_2}$$

we have that

$$u^* = 1, \quad v^* = \frac{k_1 - k_2 a}{a}$$

is a positive steady state and

$$h_u(u^*, v^*) = \frac{k_2}{k_1} (k_1 - k_2 a) > 0, \quad h_v(u^*, v^*) = \frac{a(k_2 a - k_1)}{k_1} < 0$$

$$\begin{aligned} h_v(u^*, v^*) + \chi u^* h_u(u^*, v^*) &= \frac{a(k_2 a - k_1)}{k_1} + \frac{k_2 \chi}{k_1} (k_1 - k_2 a) \\ &= \frac{(k_1 - k_2 a) k_2}{k_1} \left( \chi - \frac{a}{k_1 k_2} \right) > 0. \end{aligned}$$

$$g_u(u^*, v^*) h_v(u^*, v^*) - g_v(u^*, v^*) h_u(u^*, v^*) = -\mu \frac{a(k_2 a - k_1)}{k_1} > 0.$$

We have that assumptions (5)-(9) are satisfied, therefore for any  $\lambda_n \in \Sigma$ , there exists

$$\chi_n := \frac{\mu a + a \lambda_n}{\lambda_n k_1}$$

and a continuous component of solutions  $(u_n(\chi), v_n(\chi), \chi)$  such that  $(u^*, v^*, \chi_n) \in (u_n(\chi), v_n(\chi), \chi)$  and  $(u(\chi), v(\chi), \chi)$  is a nontrivial solution for  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$ ,  $\chi \neq \chi_n$ .

Example 2. Let us consider for problem (2) the  $C^2(\mathbb{R}^2)$  class functions  $h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  given by

$$h(u, v) = f_1(u) - v \tag{23}$$

and

$$g(u, v) = -\chi uv + f_2(u) \tag{24}$$

where  $f_{1,2} \in C^2(\mathbb{R})$  are two chosen functions.  $h$  and  $g$  satisfy,

$$\chi f_1(s) + \chi s f_1'(s) - f_2'(s) \neq 0, \quad \text{for any } s > 0 \tag{25}$$

$$f_1'(s) > \frac{1}{\chi s}, \quad \text{for any } s > 0 \tag{26}$$

$$\chi s f_1'(s) - f_2'(s) > 0, \quad \text{for any } s > 0 \tag{27}$$

and there exists  $s^* > 0$  such that

$$\chi s^* f_1(s^*) - f_2(s^*) = 0. \tag{28}$$

Therefore hypothesis (5)-(9) are verified.

Dynamical system (10) presents the steady states  $(u^*, v^*)$  defined implicitly by

$$v^* = f_1(u^*),$$

and

$$\chi u^* f_1(u^*) = f_2(u^*).$$

In view of (25), the function  $l(s) := \chi s f_1(s) - f_2(s)$  is a monotone function for  $s > 0$  and thanks to (28) there exists a unique  $u^*$ . Thanks to Theorem 2.1 we conclude that for any  $\lambda_n \in \Sigma$ , there exists  $\chi = \chi_n$

$$\chi_n = \frac{\mu a + a \lambda_n}{\lambda_n k_1}$$

and a continuous component of solutions  $(u_n(\chi), v_n(\chi), \chi)$  such that  $(u^*, v^*, \chi_n) \in (u_n(\chi), v_n(\chi), \chi)$  and  $(u(\chi), v(\chi), \chi)$  is a nontrivial solution for  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$ ,  $\chi \neq \chi_n$ .

Example 3. A particular case to solve problem (2) is considering the functions  $h$  and  $g$  defined as follows

$$h(u, v) = \frac{\alpha}{\chi} \ln(u) - v, \quad \text{with } \alpha > 1 \quad (29)$$

and

$$g(u, v) = u(\beta - u - \chi v), \quad \text{with } \beta < 2, \quad (30)$$

for  $\alpha$  and  $\beta$  satisfying

$$\alpha > \beta.$$

We notice that

$$\begin{aligned} \frac{\partial h}{\partial v} &= -1, \\ \frac{\partial h}{\partial v} + \chi u \frac{\partial h}{\partial u} &= \alpha - 1 > 0, \quad \forall \alpha > 1, \end{aligned}$$

and

$$g_u h_v - g_v h_u = -(\beta - 2u - \chi v) + \chi u \frac{\alpha}{\chi u} = 2u + \chi v + \alpha - \beta > 0,$$

for  $\alpha > \beta$ . In this case, the steady states  $(u^*, v^*)$  of the dynamical system (10) satisfy

$$\begin{cases} v^* = \frac{\alpha}{\chi} \ln u^*, \\ v^* = \beta - u^*, \end{cases} \quad (31)$$

and in view of monotonicity of the function

$$l(s) := s + \frac{\alpha}{\chi} \ln s - \beta$$

for  $s > 0$  we have the existence of a unique steady state  $u^*$  defined explicitly by  $l(u^*) = 0$  and  $v^* = \beta - u^*$ .

As the the previous examples, we conclude that for any  $\lambda_n \in \Sigma$ , there exists  $\chi_n$  defined by

$$\chi_n := \frac{\mu a + a \lambda_n}{\lambda_n k_1}$$

and a continuous component of solutions  $(u_n(\chi), v_n(\chi), \chi)$  such that  $(u^*, v^*, \chi_n) \in (u_n(\chi), v_n(\chi), \chi)$  and  $(u(\chi), v(\chi), \chi)$  is a nontrivial solution for  $\chi \in (\chi_n - \epsilon, \chi_n + \epsilon)$ ,  $\chi \neq \chi_n$ .



**3. Evolution problem for the limit case**  $h_v + \chi u h_u = 0$ ,  $g = 0$ . In this section we consider the case II, where  $h$  satisfies

$$h_v + \chi u h_u = 0 \tag{32}$$

for

$$g \equiv 0. \tag{33}$$

(32) is a linear first order partial differential equation whose solution is given by

$$h(u, v) := \tilde{h}(ue^{-\chi v}), \tag{34}$$

where  $\tilde{h} \in C^1(\mathbb{R})$  is any function satisfying

$$\tilde{h}(w^*) = 0, \quad \text{for } w^* = u^* e^{-\chi v^*} > 0. \tag{35}$$

For simplicity in the notation we drop the tilde and consider  $h = h(ue^{-\chi v})$ . We also assume that there exists  $\epsilon > 0$  such that

$$-h - ue^{-\chi v} h' < 0, \quad \text{for } |ue^{-\chi v} - w^*| \leq \epsilon. \tag{36}$$

Assumption (36) is satisfied, for instance by

$$h(ue^{-\chi v}) = \ln(u) - \chi v - \ln(w^*), \quad \text{for any } \epsilon < e^{-w^*}$$

or by

$$h(ue^{-\chi v}) = ue^{-\chi v} - w^*, \quad \text{for any } \epsilon < w^*$$

for a range of  $\chi$ .

Under assumptions (32) and (33), system (2) becomes

$$\begin{cases} u_t - \Delta u = -\text{div}(\chi u \nabla v), & \text{in } \Omega \times (0, T), \\ v_t = h(ue^{-\chi v}), & \text{in } \Omega \times (0, T), \\ \frac{\partial u}{\partial n} = 0, & \text{in } \partial\Omega, \\ u(0, x) = u_0(x), & \text{in } \Omega, \\ v(0, x) = v_0(x), & \text{in } \Omega \end{cases} \tag{37}$$

where the initial data satisfy

$$u_0, v_0 \in \mathcal{C}^{2+\alpha}(\Omega), \tag{38}$$

for some  $\alpha \in (0, 1)$ . Then, there exist positive constants  $0 < \underline{u}_0 \leq \bar{u}_0$  such that

$$0 < \underline{u}_0 \leq u_0 \leq \bar{u}_0 < \infty, \quad x \in \Omega, \tag{39}$$

$$\frac{\partial u_0}{\partial n} = \frac{\partial v_0}{\partial n} = 0, \quad \text{in } \partial\Omega. \tag{40}$$

Notice that, under assumption (40), the solution satisfies the boundary condition

$$\frac{\partial u}{\partial n} - \chi u \frac{\partial v}{\partial n} = 0. \tag{41}$$

We introduce the change of unknown

$$u = e^{\chi v} w$$

to obtain

$$\begin{aligned} h(ue^{-\chi v}) &= h(w), \\ u_t &= e^{\chi v} w_t + \chi v_t e^{\chi v} w = e^{\chi v} w_t + \chi e^{\chi v} w h'(w) \\ -\Delta u &= -\chi \text{div}(e^{\chi v} w \nabla v) - e^{\chi v} \Delta w - e^{\chi v} \chi \nabla v \nabla w. \end{aligned}$$

System (37) becomes,

$$\begin{cases} w_t - \Delta w - \chi \nabla v \nabla w = -\chi w h(w), & \text{in } \Omega \times (0, T), \\ v_t = h(w), & \text{in } \Omega \times (0, T), \\ \frac{\partial w}{\partial n} = 0, & \text{in } \partial\Omega, \\ w(0, x) = w_0(x), & \text{in } \Omega, \\ v(0, x) = v_0(x), & \text{in } \Omega. \end{cases} \quad (42)$$

In order to prove the global existence of solutions, we first obtain some a priori bounds in  $L^\infty(\Omega)$ . The  $L^\infty(\Omega)$  result is enclosed in Lemma 3.2, where the rectangle method is applied (see Negreanu and Tello [33] for more details). To apply the method, we first need some estimates enclosed in the following lemma.

**Lemma 3.1.** *Under the assumption (32)-(33) and (12)-(15), if  $(w, v)$  is a solution of (42), then*

$$|v_{x_i}|^2 \leq \int_0^t e^{t-\tau} |h'|^2 |w_{x_i}|^2 d\tau + e^t |v_{0x_i}|^2.$$

*Proof.* We derivate the equation  $v_t = h(w)$  respect to  $x_i$  to obtain

$$v_{x_i t} = h' w_{x_i};$$

multiplying the above identity by  $v_{x_i}$  and thanks to Young inequality, we have

$$\frac{d}{dt} |v_{x_i}|^2 \leq |h'|^2 |w_{x_i}|^2 + |v_{x_i}|^2.$$

After integration in time, it results

$$|v_{x_i}|^2 \leq \int_0^t e^{t-\tau} |h'|^2 |w_{x_i}|^2 d\tau + e^t |v_{0x_i}|^2,$$

which ends the proof.  $\square$

**Lemma 3.2.** *Under the assumption (32)-(33) and (12)-(15), if  $(w, v)$  is a solution of (42) and  $w_0 = u_0 e^{-\chi v_0}$  satisfies*

$$|w_0 - w^*| < \epsilon, \quad x \in \Omega$$

for  $\epsilon$  defined in (36), there exist spatially homogeneous functions  $w_1, w_2 \in C^1([0, \infty))$  satisfying

$$0 < w_1(0) \leq w_0(x) \leq w_2(0) < \infty$$

such that

$$0 < w_1(t) \leq w^* \leq w_2(t) < \infty, \quad t \geq 0,$$

$$|w_i(t) - w^*| \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (\text{for } i = 1, 2)$$

and

$$w_1(t) \leq w(x, t) \leq w_2(t), \quad \text{for } t \geq 0, x \in \Omega. \quad (43)$$

*Proof.* First, we notice that in the first equation of (42), the righthand-side term  $-\chi wh(w)$  is independent of  $v$  and thanks to assumption (36)

$$\frac{d}{dw}(-\chi wh(w)) = -\chi(h - wh') < 0;$$

then, the solution to the equation

$$\begin{cases} \bar{w}' = -\chi\bar{w}h(\bar{w}) \\ \bar{w}(0) = \bar{w}_0 \end{cases} \tag{44}$$

(for a given constant  $\bar{w}_0$ ) is also a solution to

$$\bar{w}_t - \Delta\bar{w} - \chi\nabla v\nabla\bar{w} = -\chi\bar{w}h(\bar{w}).$$

Now, we consider “ $\bar{w}$ ”, the solution to (44) with initial datum satisfying

$$\bar{w}_0 \geq \max\{\|w_0\|_{L^\infty(\Omega)}, w^*\}, \tag{45}$$

and the new variable

$$W = w - \bar{w},$$

verifying the following equation:

$$W_t - \Delta W - \chi\nabla v\nabla W = -\chi wh(w) + \chi\bar{w}h(\bar{w}).$$

We multiply by  $W_+$  and integrate by parts to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} W_+^2 + \int_{\Omega} |\nabla W_+|^2 + \int_{\Omega} W \nabla v \nabla W_+ \leq \int_{\Omega} -[\chi h(\xi) + \xi h'(\xi)] W_+^2. \tag{46}$$

Since

$$\int_{\Omega} W_+ \nabla v \nabla W \leq \frac{1}{2} \int_{\Omega} W_+^2 |\nabla v|^2 + \frac{1}{2} \int_{\Omega} |\nabla W_+|^2$$

and thanks to Lemma 3.1 we get

$$\int_{\Omega} W_+^2 |\nabla v|^2 = \int_{\Omega} W_+^2 \left[ \int_0^t e^{t-\tau} |h'|^2 |\nabla w|^2 dx d\tau + e^t |\nabla v_0|^2 \right].$$

In view of (3) and (13) we have

$$\int_{\Omega} W_+^2 |\nabla v|^2 \leq k \int_{\Omega} W_+^2 \int_0^t e^{t-\tau} |\nabla W|^2 dx d\tau + e^t k \int_{\Omega} W_+^2,$$

then

$$\int_{\Omega} W_+^2 |\nabla v|^2 \leq k \|W_+\|_{L^\infty(\Omega)}^2 \int_0^t \int_{\Omega} e^{t-\tau} |\nabla W|^2 dx d\tau + e^t k \int_{\Omega} W_+^2.$$

Therefore, inequality (46) becomes

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[ \int_{\Omega} W_+^2 + \int_0^t \int_{\Omega} |\nabla W_+|^2 \right] \\ & \leq k \|W_+\|_{L^\infty(\Omega)}^2 \int_0^t \int_{\Omega} e^{t-\tau} |\nabla W|^2 dx d\tau + k(1 + e^t) \int_{\Omega} W_+^2. \end{aligned}$$

In view of (45), and thanks to Gronwall's Lemma, we obtain, for any  $T < \infty$

$$w \leq \bar{w}, \quad \text{for } t < T$$

and taking limits when  $T \rightarrow \infty$  we get

$$w \leq \bar{w}, \quad \text{for } t < \infty. \tag{47}$$

In the same way, we prove that

$$w \geq \underline{w}, \quad \text{for } t < \infty \tag{48}$$

for  $\underline{w}$  satisfying

$$\begin{cases} \underline{w}' = -\chi \underline{w} h(\underline{w}) \\ \underline{w}(0) = \underline{w}_0 \end{cases} \quad (49)$$

and

$$\underline{w}_0 = \min\{\inf_{x \in \Omega} \{w_0\}, w^*\}.$$

We observe that, under assumptions (35), (36), the solution to the equation

$$\begin{cases} y' = -\chi y h(y) \\ y(0) = y_0 \end{cases} \quad (50)$$

satisfies

$$y(t) \rightarrow w^*$$

for any  $y_0 > 0$  and the proof ends.  $\square$

Notice that the boundedness of  $w$  does not guarantee the boundedness of  $u$  since the uniform boundedness of  $v$  is not obtained. Nevertheless, by integration we obtain

$$\|v\|_{L^\infty(\Omega)} \leq c_0(1+t). \quad (51)$$

**Theorem 3.3.** *Under the assumption (32)-(33) and (12)-(15), there exists a unique global solution  $(w, v)$  of (42), with*

$$w, v \text{ in } C_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_\infty). \quad (52)$$

The proof reproduces the steps of Theorem 2.2 in [10] using Lemma 3.2 and (51) instead of Theorem 2.1 in [10].

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