ON A TWO SPECIES CHEMOTAXIS MODEL WITH SLOW CHEMICAL DIFFUSION*

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Abstract. In this paper we consider a system of three parabolic equations modeling the behavior of two biological species moving attracted by a chemical factor. The chemical substance verifies a parabolic equation with slow diffusion. The system contains second order terms in the first two equations modeling the chemotactic effects. We apply an iterative method to obtain the global existence of solutions using that the total mass of the biological species is conserved. The stability of the homogeneous steady states is studied by using an energy method. A final example is presented to illustrate the theoretical results.

Key words. chemotaxis, global existence, Moser–Alikakos iteration, asymptotic behavior, stability

AMS subject classifications. 35A01, 35B40, 35B35, 35K57, 95D25

DOI. 10.1137/140971853

1. Introduction. *Chemotaxis* is the capacity of organisms to move along a chemical gradient. Such movement, maybe towards or away from a higher concentration of the chemical substance, has been investigated by different authors, not only from a biological point of view but also from mathematical, physical, or chemical perspectives, among others. In particular, in the early 1970s, Keller and Segel, proposed a mathematical model of two parabolic equations to describe the aggregation of Dictyostelium discoideum, a soil-living amoeba. Chemotactic abilities are crucial in many biological phenomena, such as immune system response, embryo development, tumor growth, etc. Recent studies also describe macroscopic processes in terms of chemotaxis, such as population dynamics or gravitational collapse, among others.

After the pioneering works of Keller and Segel, mathematical models of chemotaxis have been used to model the mentioned phenomena where one or several species respond to chemical stimuli. Among the mathematical challenges that the problem presents, the blowing up and the global existence of solutions, have attracted the attention of the mathematical community. The system describing the evolution of one biological species where the chemical is modeled by a second order elliptic equation has been largely studied, as the fully parabolic system; see, for instance, the review by Horstmann [12] and reference therein.

Motivated by empirical biological data, multispecies chemotaxis systems have been proposed over the last 30 years; see, for instance, Alt [1], Fasano, Mancini, and Primicerio [10], Wolansky [26], or Horstmann [13].

Parabolic-parabolic-elliptic systems, where the evolution of two biological species is described by parabolic equations and an elliptic equation models the behavior of the chemoattractant substance, have been recently analyzed by several authors. In

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Tello and Winkler [22] and Stinner, Tello, and Winkler [20] global asymptotic stability of homogeneous steady states is studied under the effects of competition between the species. The blow-up phenomenon in a bounded domain, when the interaction between the biological species is reduced to the chemical production, has been considered in Espejo, Stevens, and Velázquez [7] and [8] for simultaneous and nonsimultaneous blow-up in $\mathbb{R}^2$. See also Biler, Espejo, and Guerra [3], Biler and Guerra [4] for bounded domains and Conca and Espejo [5], and Conca, Espejo, and Vilches [6] for the two-dimensional case in the whole space (see also Espejo and Wolanski [9] for more details and cases).

The one species fully parabolic system with signal-dependent chemotactic sensitivities has been already researched in the literature. In Biler [2], the two-dimensional case was studied, the author proving global existence of solutions for $\chi(w) = 1/w$ with initial data satisfying $\nabla u_0 \in [L^2(\omega)]^2$. In Winkler [24] and [25], the global existence is obtained for arbitrary dimension $n \geq 2$ for $\chi(w) = \chi_0/w$ with a positive constant $\chi_0 < \sqrt{2/n}$. In [21], Stinner and Winkler investigated the existence of week solutions for arbitrary large chemotactic sensitivity χ_0.

Signal dependent chemotactic sensitivities appear in many other PDE systems, such as the parabolic-elliptic case, systems coupled with Stokes or Navier–Stokes equations modeling different biological phenomena (see, for instance, [14] and [15]).

The parabolic-parabolic-ODE problem, has been recently studied in Negreanu and Tello [18], where the rectangles method used in Friedman and Tello [11] (see also [17]) for the parabolic-ODE problem cannot be applied due to the second component of the species. In Negreanu and Tello [16], the logistic growth term drives the solution to the unique positive constant stationary state under some structural restrictions in the chemical stimuli.

In this paper we focus on the fully parabolic problem, where the diffusion coefficient of the chemical substance is strictly less than 1. The system has been previously considered by Horstmann [13] where the linear stability is studied for a range of parameters with constant chemoattractant sensitivity and linear chemical production.

We denote the densities of the species by $u(x,t)$ and $v(x,t)$, the concentration of the chemoattractant by $w(x,t)$, Ω is a bounded and regular domain of $\mathbb{R}^n$ for $n \geq 1$. The fully parabolic system is given by the following system of equations:

$$(1.1) \quad \begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w), & x \in \Omega, \ t > 0, \\ w_t = \varepsilon \Delta w + h(u,v,w), & x \in \Omega, \ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega, \ t > 0 \end{cases}$$

for $\varepsilon > 0$, and initial data

$$(1.2) \quad u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(0,x) = w(x), \quad x \in \Omega.$$

In (1.1), $h(u,v,w)$ denotes the balance between the production and degradation of the chemical which depends explicitly on the living organisms. The chemotactic sensitivity coefficients χ_1 and χ_2 are assumed to be positive and regular, i.e.,

$$(1.3) \quad \chi_i \in W^{1,\infty}_{loc}(\mathbb{R}_+) \cap C^1(\mathbb{R}_+), \quad \chi_i > 0 \quad \text{for } i = 1, 2.$$

For technical reasons we also assume that

$$(1.4) \quad \chi_i' + \frac{1}{1-\varepsilon}\chi_i^2 \leq 0 \quad \text{for } i = 1, 2.$$

We consider that the balance between production and degradation of a chemical is a regular function

$$\text{(1.5)} \qquad h \in W^{1,\infty}(\mathbb{R}_+^3) \cap C^1(\mathbb{R}_+^3),$$

monotone increasing in u and v and monotone decreasing in w, i.e.,

$$\text{(1.6)} \qquad \frac{\partial h}{\partial u} \geq \epsilon_u > 0 \quad \text{and} \quad \frac{\partial h}{\partial v} \geq \epsilon_v > 0,$$

$$\text{(1.7)} \qquad \frac{\partial h}{\partial w} \leq -\epsilon_w < 0$$

for some positive constants ϵ_u, ϵ_v, and ϵ_w.

We denote by u^* and v^* the positive constants defined as

$$\text{(1.8)} \qquad \begin{aligned} u^* &:= \frac{1}{|\Omega|} \int_\Omega u_0(x)dx = \frac{1}{|\Omega|} \int_\Omega u(x,t)dx, \\ v^* &:= \frac{1}{|\Omega|} \int_\Omega v_0(x)dx = \frac{1}{|\Omega|} \int_\Omega v(x,t)dx. \end{aligned}$$

Thanks to assumption (1.7) and the implicit function theorem we may deduce the existence of a unique constant w^* satisfying

$$\text{(1.9)} \qquad h(u^*, v^*, w^*) = 0.$$

In section 2 we study the global existence of solutions to (1.1)–(1.2) and obtain global uniform boundedness of the solutions under the following constrains in h and χ_i (for $i = 1, 2$):

- There exist two constants $\underline{w}$ and $\overline{w}$ verifying

$$\text{(1.10)} \qquad 0 \leq \underline{w} < w_0 < \overline{w}, \qquad \underline{w} < w^* < \overline{w}.$$

- There exist some positive constants k_1 and k_2 such that

$$\text{(1.11)} \qquad -h(0,0,w) \leq \frac{k_i}{\chi_i(w)} \quad \text{for } i = 1, 2 \quad \text{and} \quad \underline{w} \leq w \leq \overline{w}.$$

- There exist k_{01} and k_{02} positive constants such that

$$\text{(1.12)} \qquad 0 < k_{0i} \leq \chi_i(w)e^{\int_{\underline{w}}^w \chi_i(s)ds} \qquad \text{for } i = 1, 2 \text{ and} \quad w \geq \underline{w}.$$

- We also assume

$$\text{(1.13)} \quad h(u,v,\underline{w}) \geq 0, \qquad h(u,v,\overline{w}) < 0 \quad \text{for} \quad 0 \leq u \leq \overline{u}, \quad 0 \leq v \leq \overline{v},$$

where

$$\text{(1.14)} \qquad \overline{u} := f_{1\infty}(\overline{w}) \max \left\{ k_1 \left(\epsilon_u k_{01}(1 - \varepsilon) \right)^{-1}, \|u_0\|_{L^\infty(\Omega)} \right\}$$

and

$$\text{(1.15)} \qquad \overline{v} := f_{2\infty}(\overline{w}) \max \left\{ k_2 \left(\epsilon_v k_{02}(1 - \varepsilon) \right)^{-1}, \|v_0\|_{L^\infty(\Omega)} \right\}$$

for $f_{1\infty}$ and $f_{2\infty}$ defined by

$$\text{(1.16)} \qquad f_{i\infty}(\overline{w}) = exp \left\{ \frac{1}{1 - \epsilon} \int_{\underline{w}}^{\overline{w}} \chi_i(s)ds \right\} \qquad \text{for } i = 1, 2.$$

Under assumptions (1.11)–(1.16) we have the uniform boundedness of the solutions, more precisely, we obtain the bounds

$$0 \le u(x,t) \le \overline{u}, \quad 0 \le v(x,t) \le \overline{v}, \quad \text{and} \quad \underline{w} \le w(x,t) \le \overline{w} \quad \text{for } t > 0, \quad x \in \Omega.$$

That result, together with the existence of solutions is enclosed in Theorem 3.1. To achieve the uniform bounds we implement an iterative method based on the L^p-norm of the solutions inspired by the Moser–Alikakos iteration.

In section 3 without loss of generality we analyze the stability of the system for a linear profile h defined by

$$h(u,v,w) = au + v - 2\mu w \quad \text{for } a, \mu > 0.$$

By using an energy method, in Theorem 4.1 we prove the asymptotic stability of the homogeneous steady state (u^*, v^*, w^*) defined by (1.8), (1.9) under the additional restrictions

$$(1.17) \quad u\chi_1(w)\frac{\partial h}{\partial u} + v\chi_2(w)\frac{\partial h}{\partial v} + \frac{\partial h}{\partial w} < 0, \qquad 0 \le u \le \overline{u}, \quad 0 \le v \le \overline{v}, \quad \underline{w} \le w \le \overline{w},$$

and

$$C(\Omega) \max\left\{ (\chi_1(\underline{w})\overline{u})^2, \ (\chi_2(\underline{w})\overline{v})^2 \right\} \max\{1, a^2\} < 2\mu\varepsilon,$$

where $C(\Omega)$ is the smallest positive constant satisfying

$$\int_\Omega u^2 dx \le C(\Omega) \int_\Omega |\nabla u|^2 dx$$

for any $u \in H^1(\Omega)$, such that $\int_\Omega u(x)dx = 0$.

In the last section we illustrate the results obtained in the previous sections with a particular example, where the chemosensitivity functions are $\chi_i = \alpha_i/(\beta_i + w)$ for some positive values α_i, β_i $(i = 1, 2)$.

A simple case of the above example is given for the parameters

$$a = \mu = 1, \qquad \varepsilon := \frac{1}{2}, \qquad \alpha_1 = \alpha_2 = \frac{1}{2}, \qquad \beta_1 = \beta_2 = \frac{1}{16},$$

i.e.,

$$h(u,v,w) = u + v - 2w, \qquad \chi_1(w) = \chi_2 = \frac{8}{16w + 1}$$

with

$$\underline{w} = 0, \qquad \overline{w} < \frac{1}{8}.$$

Hypothesis (1.11) is verified taking

$$k_1 = k_2 = 1,$$

and (1.12) is reduced to

$$k_{01} = k_{02} = \frac{2}{(1 + 16\overline{w})^{\frac{1}{2}}}.$$

The rest of the assumptions are simple computations. Then, for any initial data (u_0, v_0, w_0) satisfying

$$\|u_0\|_{L^\infty(\Omega)} \leq \frac{\overline{w}}{(\frac{1}{16} + \overline{w})^{\frac{1}{3}}}, \|v_0\|_{L^\infty(\Omega)} \leq \frac{\overline{w}}{(\frac{1}{16} + \overline{w})^{\frac{1}{3}}}, \|w_0\|_{L^\infty(\Omega)} \leq \overline{w},$$

the solution is uniformly bounded in $(0, \infty)$ and

$$\lim_{t \to \infty} \int_\Omega |u - u^*|^2 dx = \lim_{t \to \infty} \int_\Omega |v - v^*|^2 dx = \lim_{t \to \infty} \int_\Omega |w - w^*|^2 dx = 0,$$

where the stationary solution of system (1.1) is

$$(u^*, v^*, w^*) := \left(\int_\Omega u_0(x)dx, \int_\Omega v_0(x)dx, \int_\Omega \frac{(u_0 + v_0)dx}{2} \right).$$

For the sake of simplicity we assume thorough the article that $|\Omega| = 1$ and denote by $\Omega_T = \Omega \times (0, T)$, $\Omega_\infty = \Omega \times (0, \infty)$.

2. Steady states. The steady states of problem (1.1) satisfy the system

(2.1)
$$\begin{cases} 0 = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w), & x \in \Omega, \ t > 0, \\ 0 = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w), & x \in \Omega, \ t > 0, \\ 0 = \varepsilon \Delta w + h(u, v, w), & x \in \Omega, \ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega. \end{cases}$$

The biological meaningful solutions should be positive, hence we only consider non-negative bounded steady states.

LEMMA 2.1. *Under assumptions* (1.3), (1.5)–(1.17), *for every* $\varepsilon > 0$, *any non-negative bounded solutions of* (2.1) *are constant.*

Proof. We introduce the change of variables $\tilde{u}$ and $\tilde{v}$ defined by

(2.2)
$$u = \tilde{u} \exp \left\{ \int_{\underline{w}}^w \chi_1(s)ds \right\}, \qquad v = \tilde{v} \exp \left\{ \int_{\underline{w}}^w \chi_2(s)ds \right\}.$$

Since

$$\nabla u = (\chi_1(w)\tilde{u}\nabla w + \nabla\tilde{u}) \exp \left\{ \int_{\underline{w}}^w \chi_1(s)ds \right\}$$

and

$$\Delta u = \nabla \cdot \left(\chi_1(w)u\nabla w + \exp \left\{ \int_{\underline{w}}^w \chi_1(s)ds \right\} \nabla\tilde{u} \right),$$

the first equation in (2.1) becomes

$$-\nabla \cdot \left(\exp \left\{ \int_{\underline{w}}^w \chi_1(s)ds \right\} \nabla\tilde{u} \right) = 0, \qquad x \in \Omega,$$

and the boundary condition

$$\nabla\tilde{u} \cdot \nu = 0$$

implies that $\tilde{u}$ is a constant. In the same way we obtain that $\tilde{v}$ is also a constant.

Since $\varepsilon > 0$, we have that w satisfies

$$(2.3) \qquad -\varepsilon \Delta w = h\left(\tilde{u}\exp\left\{\int_{\underline{w}}^{w}\chi_1(s)ds\right\}, \tilde{v}\exp\left\{\int_{\underline{w}}^{w}\chi_2(s)ds\right\}, w\right),$$

where $\tilde{u}$ and $\tilde{v}$ are constant. Thanks to assumption (1.17), we obtain that

$$\frac{d}{dw}h(\tilde{u}f_1(w), \tilde{v}f_2(w), w) = u\chi_1(w)h_u + v\chi_2(w)h_v + h_w < 0$$

which proves the existence of, at most one solution of (2.3). By the implicit function theorem, there exists a constant w^* satisfying $h(u^*, v^*, w^*) = 0$, which ends the proof. $\square$

3. Global existence. In this section we study the global existence of solutions and we obtain global uniform bounds in $L^\infty(\Omega)$. The result is enclosed in the following Theorem.

THEOREM 3.1. *Under* (1.3)–(1.17), (1.11)–(1.16) *for any initial data* (u_0, v_0, w_0) $\in (C^1(\overline{\Omega}))^3$ *satisfying Neumann boundary conditions and* $u_0 \geq 0$, $v_0 \geq 0$, *there exists a unique solution to* (1.1)–(1.2)

$$u, v, w \in [L^p(0, T : W^{2,p}(\Omega)) \cap W^{1,p}(0, T : L^p(\Omega))]^3$$

for $p > n$ *such that*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \leq C < \infty.$$

We divide the proof of the theorem into several steps. To obtain some a priori bounds, we need some technical lemmas.

LEMMA 3.2. *Under assumption* (1.5), *any solution to* (1.1)–(1.2) *satisfies*

$$\int_\Omega u dx = \int_\Omega u_0 dx, \qquad \int_\Omega v dx = \int_\Omega v_0 dx,$$

$$\int_\Omega w dx \leq C\left(\|h\|_{W^{1,\infty}(\mathbb{R}_+^3)}, \epsilon_w, \int_\Omega u_0 dx, \int_\Omega v_0 dx, \int_\Omega w_0 dx, w^*\right).$$

Proof. We integrate the first two equation in (1.1) to obtain the mass conservation of the species, i.e.,

$$\frac{d}{dt}\int_\Omega u dx = 0, \qquad \frac{d}{dt}\int_\Omega v dx = 0$$

which prove $\int_\Omega u dx = \int_\Omega u_0 dx$ and $\int_\Omega v dx = \int_\Omega v_0 dx$. Integrating the third equation of (1.1) we obtain

$$\frac{d}{dt}\int_\Omega w dx = \int_\Omega h(u, v, w)dx;$$

by the mean value theorem, we have that

$$h(u, v, w) = \frac{\partial h}{\partial u}\bigg|_{u=\xi_1}(u - u^*) + \frac{\partial h}{\partial v}\bigg|_{v=\xi_2}(v - v^*) + \frac{\partial h}{\partial w}\bigg|_{w=\xi_3}(w - w^*)$$

and thanks to (1.6) and (1.7)

$$\int_\Omega h(u,v,w)dx \le \|h\|_{W^{1,\infty}(\mathbb{R}^3_+)} \int_\Omega (u+v+w^*)dx - \epsilon_w \int_\Omega wdx$$

and then

$$\frac{d}{dt}\int_\Omega wdx + \epsilon_w \int_\Omega wdx \le \|h\|_{W^{1,\infty}(\mathbb{R}^3_+)} \int_\Omega (u+v+w^*)dx.$$

Applying the maximum principle we achieve the results.   □

LEMMA 3.3. *Given* $(u_0, v_0, w_0) \in [C^1(\overline{\Omega})]^3$ *positive initial data, under assumptions* (1.3)–(1.5) *for* $\varepsilon > 0$, *there exists* $T > 0$ *small enough and a unique solution* (u, v, w) *to* (1.1) *in* Ω_T *satisfying*

$$u, v, w \in [L^p(0, T : W^{2,p}(\Omega)) \cap W^{1,p}(0, T : L^p(\Omega))]^3$$

for $p > n$. *Moreover we have*

$$u \ge 0, \quad v \ge 0, \quad w \ge \underline{w}.$$

Proof. We introduce the following functions:

(3.1) $$f_{1\infty}(w) = e^{\frac{1}{1-\varepsilon}\int_{\underline{w}}^w \chi_1(s)ds}, \qquad f_{2\infty}(w) = e^{\frac{1}{1-\varepsilon}\int_{\underline{w}}^w \chi_2(s)ds},$$

and the new variables $\tilde{u}$ and $\tilde{v}$ given by

(3.2) $$u = f_{1\infty}(w)\tilde{u}, \qquad v = f_{2\infty}(w)\tilde{v}.$$

Operating, we have

$$u_t = f'_{1\infty}(w)w_t\tilde{u} + f_{1\infty}(w)\tilde{u}_t, \qquad v_t = f'_{2\infty}(w)w_t\tilde{v} + f_{2\infty}(w)\tilde{v}_t,$$
$$\nabla u = f'_{1\infty}(w)\tilde{u}\nabla w + f_{1\infty}(w)\nabla\tilde{u}, \qquad \nabla v = f'_{2\infty}(w)\tilde{v}\nabla w + f_{2\infty}(w)\nabla\tilde{v},$$
$$\Delta u = f''_{1\infty}\tilde{u}|\nabla w|^2 + f'_{1\infty}(w)\nabla\tilde{u}\nabla w + f'_{1\infty}(w)\tilde{u}\Delta w + \nabla\cdot(f_{1\infty}(w)\nabla\tilde{u}),$$
$$\Delta v = f''_{2\infty}\tilde{v}|\nabla w|^2 + f'_{2\infty}(w)\nabla\tilde{v}\nabla w + f'_{2\infty}(w)\tilde{v}\Delta w + \nabla\cdot(f_{2\infty}(w)\nabla\tilde{v}),$$
$$\nabla\cdot(u\chi_1(w)\nabla w) = \tilde{u}f_{1\infty}(w)\chi_1(w)\Delta w + \tilde{u}(f'_{1\infty}(w)\chi_1(w) + f_{1\infty}(w)\chi'_1(w))|\nabla w|^2$$
$$\qquad\qquad + f_{1\infty}(w)\chi_1(w)\nabla\tilde{u}\nabla w,$$
$$\nabla\cdot(v\chi_2(w)\nabla w) = \tilde{v}f_{2\infty}(w)\chi_2(w)\Delta w + \tilde{v}(f'_{2\infty}(w)\chi_2(w) + f_{2\infty}(w)\chi'_2(w))|\nabla w|^2$$
$$\qquad\qquad + f_{2\infty}(w)\chi_2(w)\nabla\tilde{v}\nabla w,$$

where

$$f'_{i\infty}(w) = \frac{1}{1-\varepsilon}f_{i\infty}(w)\chi_i(w), \quad f''_{i\infty}(w) = \frac{f_{i\infty}(w)}{(1-\varepsilon)^2}(\chi_i^2(w) + (1-\varepsilon)\chi'_i(w)), \quad i = 1,2.$$

Considering the following operators

$$\mathcal{L}_1(w)\tilde{u} = \tilde{u}_t - \Delta\tilde{u} - \frac{1+\varepsilon}{1-\varepsilon}\chi_1(w)\nabla w\nabla\tilde{u},$$
$$\mathcal{L}_2(w)\tilde{v} = \tilde{v}_t - \Delta\tilde{v} - \frac{1+\varepsilon}{1-\varepsilon}\chi_2(w)\nabla w\nabla\tilde{v},$$
$$g_i(\tilde{u}, \tilde{v}, w) = -\frac{1}{1-\varepsilon}f_{i\infty}(w)\chi_i(w)h(f_{1\infty}(w)\tilde{u}, f_{2\infty}(w)\tilde{v}, w),$$

system (1.1) becomes

$$(3.3) \quad \begin{cases} \mathcal{L}_1(w)\tilde{u} = \tilde{u}g_1(\tilde{u}, \tilde{v}, w) + \tilde{u}\varepsilon\frac{f_1(w)}{(1-\epsilon)^2}(\chi_1^2(w) + (1 - \varepsilon)\chi_1'(w))|\nabla w|^2, \\ \mathcal{L}_2(w)\tilde{u} = \tilde{v}g_2(\tilde{u}, \tilde{v}, w) + \tilde{v}\varepsilon\frac{f_2(w)}{(1-\epsilon)^2}(\chi_2^2(w) + (1 - \varepsilon)\chi_2'(w))|\nabla w|^2, \\ w_t = \varepsilon\Delta w + h(f_{1\infty}(w)\tilde{u}, f_{2\infty}(w)\tilde{v}, w), \qquad x \in \Omega, \ t > 0. \end{cases}$$

Considering a fixed point argument in $[L^p(0, T : W^{2,p}(\Omega)) \cap W^{1,p}(0, T : L^p(\Omega))]^2$ (for $p > n$), we take $w \in C(0, T : C^1(\overline{\Omega}))$ satisfying $w \in (\underline{w}, \overline{w})$ and $|\nabla w| < C$ and define $\tilde{u}$, $\tilde{v}$ as the unique solution to

$$\begin{cases} \mathcal{L}_1(w)\tilde{u} = \tilde{u}g_1(\tilde{u}, \tilde{v}, w) + \tilde{u}\varepsilon\frac{f_1(w)}{(1-\epsilon)^2}(\chi_1^2(w) + (1 - \varepsilon)\chi_1'(w))|\nabla w|^2, \\ \mathcal{L}_2(w)\tilde{u} = \tilde{v}g_2(\tilde{u}, \tilde{v}, w) + \tilde{v}\varepsilon\frac{f_2(w)}{(1-\epsilon)^2}(\chi_2^2(w) + (1 - \varepsilon)\chi_2'(w))|\nabla w|^2, \end{cases}$$

(for more details see Quittner and Souplet [19, Remark 48.3]). Nonnegativity of $\tilde{u}$ and $\tilde{v}$ is a consequence of the multiplicative terms $\tilde{u}$ and $\tilde{v}$ on the right-hand side part of (3.3). Notice also that $\tilde{u}$ and $\tilde{v}$ are regular functions satisfying

$$(3.4) \qquad \mathcal{L}_1(w)\tilde{u} \le \tilde{u}g_1(\tilde{u}, \tilde{v}, w), \qquad \mathcal{L}_2(w)\tilde{v} \le \tilde{v}g_2(\tilde{u}, \tilde{v}, w).$$

Thanks to (3.4) we may construct supersolutions $\overline{u}$ and $\overline{v}$ such that

$$0 \le u \le \overline{u}, \qquad 0 \le v \le \overline{v} \quad \text{ for } t < T,$$

for T small enough. We apply Theorem 2.1 of Negreanu and Tello [18] to obtain a solution in $(0, T)$. For w we solve the parabolic equation

$$w_t = \varepsilon\Delta w + h(f_{1\infty}(w)\tilde{u}, f_{2\infty}(w)\tilde{v}, w).$$

Applying the Schauder fixed point theorem, thanks to the parabolic regularity and the compact embeddings, we get the local existence of the solutions. Standard technics used in parabolic equations assert uniqueness of solutions.

To see that $w \ge \underline{w}$ we consider the equation

$$w_t - \varepsilon\Delta w = h(u, v, w) \ge h(0, 0, w),$$

where h satisfies (1.7) and (1.13) which implies

$$h(0, 0, \underline{w}) \ge 0 \quad \text{ and } \quad \left.\frac{\partial h}{\partial w}\right|_{u=v=0, w=\underline{w}} \le -\epsilon_w.$$

Thanks to the maximum principle and (1.10) we obtain that $w \ge \underline{w}$ and the proof ends. □

The solution is extended to the interval $(0, T_{\max})$, where $T_{\max}$ has the property

$$(3.5) \qquad \limsup_{t \to T_{\max}} \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + t = \infty.$$

To finish the proof, we need to introduce new notation f_{1p} and f_{2p},

$$(3.6) \qquad f_{1p} = e^{\frac{1}{1-\varepsilon\frac{p-1}{p}}\int_{\underline{w}}^w \chi_1(s)ds}, \qquad f_{2p}(w) = e^{\frac{1}{1-\varepsilon\frac{p-1}{p}}\int_{\underline{w}}^w \chi_2(s)ds}$$

with

$$\frac{d}{dw}f_{1p} = \frac{1}{1 - \varepsilon\frac{p-1}{p}}\chi_1(w)f_{1p} \quad \text{and} \quad \frac{d}{dw}f_{2p} = \frac{1}{1 - \varepsilon\frac{p-1}{p}}\chi_2(w)f_{2p}.$$

LEMMA 3.4. *Let $p > 1$ and f_{ip} (for $i = 1, 2$) defined in (3.6). Then, under assumptions (1.3)–(1.16), the following hold:*

$$(3.7) \qquad \frac{1}{p-1}\frac{d}{dt}\int_\Omega u^p f_{1p}^{1-p}dx \le -\frac{1}{1 - \varepsilon\frac{p-1}{p}}\int_\Omega u^p\chi_1(w)f_{1p}^{1-p}h(u,v,w)dx$$

and

$$(3.8) \qquad \frac{1}{p-1}\frac{d}{dt}\int_\Omega v^p f_{2p}^{1-p}dx \le -\frac{1}{1 - \varepsilon\frac{p-1}{p}}\int_\Omega v^p\chi_2(w)f_{2p}^{1-p}h(u,v,w)dx.$$

Proof. For $p > 1$

$$\frac{d}{dt}\int_\Omega u^p f_{1p}^{1-p}dx$$

$$= p\int_\Omega u^{p-1}u_t f_{1p}^{1-p}dx + \int_\Omega u^p[f_{1p}^{1-p}]'(\varepsilon\Delta w + h(u,v,w))dx$$

$$(3.9) \qquad = p\int_\Omega u^{p-1}u_t f_{1p}^{1-p}dx + \frac{1-p}{1 - \epsilon\frac{p-1}{p}}\int_\Omega u^p f_{1p}^{1-p}\chi_1(w)(\varepsilon\Delta w + h(u,v,w))dx$$

$$= p\int_\Omega u^{p-1}f_{1p}^{1-p}\left[\Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \frac{1-p}{p - \varepsilon(p-1)}u\chi_1(w)\varepsilon\Delta w\right]dx$$

$$\quad + \frac{1-p}{1 - \varepsilon\frac{p-1}{p}}\int_\Omega u^p\chi_1(w)f_{1p}^{1-p}h(u,v,w)dx.$$

Taking into account that

$$u\chi_1(w)\Delta w = \nabla\cdot(u\chi_1(w)\nabla w) - \chi_1(w)\nabla u\nabla w - u\chi_1'|\nabla w|^2$$

it gives

$$p\int_\Omega u^{p-1}f_{1p}^{1-p}\left[\Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \frac{(1-p)\varepsilon}{p - \varepsilon(p-1)}u\chi_1(w)\Delta w\right]dx$$

$$= p\int_\Omega u^{p-1}f_{1p}^{1-p}\left[\Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \frac{(1-p)\varepsilon}{p - \varepsilon(p-1)}\nabla(u\chi_1(w)\nabla w)\right]$$

$$\quad + p\int_\Omega u^{p-1}f_{1p}^{1-p}\left[-\frac{(1-p)\varepsilon}{p - \varepsilon(p-1)}(\chi_1(w)\nabla u\nabla w + u\chi_1'|\nabla w|^2)\right]dx.$$

Since

$$\Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \frac{(1-p)\varepsilon}{p - \varepsilon(p-1)}\nabla(u\chi_1(w)\nabla w) = \nabla\left(f_{1p}\nabla\frac{u}{f_{1p}}\right)$$

we have

$$I := \int_\Omega \frac{u^{p-1}}{f_{1p}^{1-p}} \left[p\Delta u - p\nabla \cdot (u\chi_1(w)\nabla w) + \frac{(1-p)\varepsilon}{1 - \varepsilon\frac{p-1}{p}} u\chi_1(w)\Delta w \right] dx$$

$$= \int_\Omega \frac{u^{p-1}}{f_{1p}^{1-p}} \left[p\Delta u - p\chi_1\nabla u\nabla w - pu\chi_1'(w)|\nabla w|^2 + \left(\frac{(1-p)\varepsilon}{1 - \varepsilon\frac{p-1}{p}} - p \right) u\chi_1(w)\Delta w \right] dx$$

$$= p\int_\Omega \frac{u^{p-1}}{f_{1p}^{p-1}} \left[\nabla\left(f_{1p}\nabla\frac{u}{f_{1p}} \right) - \frac{1-p}{p - \varepsilon(p-1)}\varepsilon(\chi_1(w)\nabla u\nabla w + u\chi_1'|\nabla w|^2) \right] dx$$

$$= p\int_\Omega \frac{u^{p-1}}{f_{1p}^{p-1}} \left[\nabla\left(f_{1p}\nabla\frac{u}{f_{1p}} \right) - \frac{1-p}{p - \varepsilon(p-1)}\varepsilon(\chi_1(w)\nabla u\nabla w + u\chi_1'|\nabla w|^2) \right] dx$$

$$= p(p-1)\int_\Omega \frac{u^{p-2}}{f_{1p}^{p-2}} \left[-f_{1p}\left|\nabla\frac{u}{f_{1p}}\right|^2 + \frac{u\varepsilon(\chi_1(w)\nabla u\nabla w + u\chi_1'|\nabla w|^2)}{f_{1p}(p - \varepsilon(p-1))} \right] dx.$$

We denote by I_1, I_2, and I_3 the terms on the right-hand side of the previous equation, i.e.,

$$I_1 := -p(p-1)\int_\Omega u^{p-2} f_{1p}^{3-p} \left|\nabla\frac{u}{f_{1p}}\right|^2 dx$$

$$I_2 := \frac{p(p-1)\varepsilon}{p - \varepsilon(p-1)} \int_\Omega u^{p-1} f_{1p}^{1-p}\chi_1(w)\nabla u\nabla w dx$$

and

$$I_3 := \frac{p(p-1)\varepsilon}{p - \varepsilon(p-1)} \int_\Omega u^{p-1} f_{1p}^{1-p}u\chi_1'|\nabla w|^2 dx.$$

We consider now I_2. Since

$$\nabla u = f_{1p}\nabla\frac{u}{f_{1p}} + \frac{f_{1p}'}{f_{1p}}u\nabla w = f_{1p}\nabla\frac{u}{f_{1p}} + \frac{1}{1 - \varepsilon\frac{p-1}{p}}\chi_1(w)u\nabla w$$

it gives

$$\int_\Omega u^{p-1} f_{1p}^{1-p}\chi_1(w)\nabla u\nabla w dx$$

$$= \int_\Omega u^{p-1} f_{1p}^{1-p}\chi_1(w) \left[f_{1p}\nabla\frac{u}{f_{1p}} + \frac{1}{1 - \varepsilon\frac{p-1}{p}}\chi_1(w)u\nabla w \right] \nabla w dx$$

which implies

$$I_2 = \frac{(p-1)p\varepsilon}{p - \varepsilon(p-1)} \int_\Omega u^{p-1} f_{1p}^{2-p}\chi_1(w)\nabla\frac{u}{f_{1p}}\nabla w$$

$$+ \frac{(p-1)p^2\varepsilon}{(p - \varepsilon(p-1))^2} \int_\Omega u^{p-1} f_{1p}^{1-p}\chi_1^2(w)u|\nabla w|^2 dx$$

$$\le p(p-1)\int_\Omega u^{p-2} f_{1p}^{3-p} \left|\nabla\frac{u}{f_{1p}}\right|^2 + \frac{\varepsilon^2 p(p-1)}{(p - \varepsilon(p-1))^2} \int_\Omega u^p f_{1p}^{1-p}\chi_1^2(w)|\nabla w|^2 dx$$

$$+ \frac{(p-1)p^2\varepsilon}{(p - \varepsilon(p-1))^2} \int_\Omega u^p f_{1p}^{1-p}\chi_1^2(w)|\nabla w|^2 dx.$$

For I_2 we have the following bound:

$$I_2 \leq p(p-1) \int_\Omega u^{p-2} f_{1p}^{3-p} |\nabla \frac{u}{f_{1p}}|^2$$
$$+ \left[\frac{\varepsilon^2 p(p-1)}{(p-\varepsilon(p-1))^2} + \frac{(p-1)p^2\varepsilon}{(p-\varepsilon(p-1))^2} \right] \int_\Omega u^p f_{1p}^{1-p} \chi_1^2(w) |\nabla w|^2 dx.$$

Recalling, $I := I_1 + I_2 + I_3$, with I_1, I_2, and I_3 defined above, it satisfies

$$(3.10) \qquad I \leq \frac{p(p-1)\varepsilon}{p-\varepsilon(p-1)} \int_\Omega u^p f_{1p}^{1-p} |\nabla w|^2 \left[\chi_1' + \frac{\varepsilon+p}{p-\varepsilon(p-1)} \chi_1^2 \right] dx.$$

Due to the function

$$g_\varepsilon(p) := \frac{\varepsilon+p}{p-\varepsilon(p-1)}$$

being monotone increasing in p for $p \geq 1$, we have that

$$\frac{\varepsilon+p}{p-\varepsilon(p-1)} \leq \frac{1}{1-\varepsilon} \qquad \text{for } p \geq 1$$

and thanks to (1.4)

$$I \leq 0 \text{ for any } p > 1.$$

As a consequence of (3.9) we prove the first inequality (3.7). In the same way we obtain (3.8) and the proof in done. □

In order to prove the global boundedness of the solution, we proceed in several steps: First, we see that, as far as $\|w(x,t)\|_{L^\infty(\Omega)} \leq \overline{w}$ we have that u and v are uniformly bounded by $\overline{u}$ and $\overline{v}$, respectively (Lemma 3.6). To that purpose we consider T^* defined by

$$(3.11) \qquad T^* := \begin{cases} T_{\max} & \text{if } w \leq \overline{w} \text{ in } (0, T_{\max}), \\ T_w & \text{if } \|w\|_{L^\infty(\Omega_{T_{\max}})} > \overline{w}, \end{cases}$$

where $T_w < T_{\max}$ satisfies that $\|w\|_{L^\infty(\Omega_{T_w})} \leq \overline{w}$ and $\|w\|_{L^\infty(\Omega_{T_w+\delta})} > \overline{w}$ for any $\delta > 0$, if $\|w\|_{L^\infty(\Omega_{T_{\max}})} > \overline{w}$.

Notice that since $w \in C(0, T_{\max} : C^1(\Omega))$ and thanks to (1.10) we have that $T^* > 0$.

LEMMA 3.5. We assume (1.3)–(1.16) and consider $p > 1$ and f_{ip} (for $i = 1, 2$) as in (3.6). Then, for any $t < T^*$, the solutions to the problem satisfy

$$\frac{1}{p-1} \frac{d}{dt} \int_\Omega u^p f_{1p}^{1-p} dx \leq -\epsilon_u k_{01} \int_\Omega u^{p+1} f_{1p}^{-p} dx + \frac{k_1}{1-\varepsilon} \int_\Omega u^p f_{1p}^{1-p} dx$$

and

$$\frac{1}{p-1} \frac{d}{dt} \int_\Omega v^p f_{2p}^{1-p} dx \leq -\epsilon_v k_{02} \int_\Omega v^{p+1} f_{2p}^{-p} dx + \frac{k_2}{1-\epsilon} \int_\Omega v^p f_{2p}^{1-p} dx$$

for k_i, k_{0i}, ϵ_u, and ϵ_v given by (1.11), (1.12), and (1.6).

Proof. We split h into several terms in the following way:

$$h(u,v,w) = h(u,v,w) - h(0,v,w) + h(0,v,w) - h(0,0,w) + h(0,0,w),$$

and thanks to (1.3), the mean value theorem, and assumptions (1.6), (1.7), and (1.11) it gives

$$-h(u,v,w) \leq -\epsilon_u u - \epsilon_v v - h(0,0,w) \leq -\epsilon_u u + \frac{k_1}{\chi_1(w)}.$$

Then, the term containing h in Lemma 3.4 is bounded in the following sense:

$$-\int_\Omega u^p \chi_1(w) f_{1p}^{1-p} h(u,v,w) dx$$

(3.12)
$$\leq -\epsilon_u \int_\Omega u^{p+1} \chi_1(w) f_{1p}^{1-p} dx + k_1 \int_\Omega u^p f_{1p}^{1-p} dx$$

$$\leq -\epsilon_u k_{01} \int_\Omega u^{p+1} f_{1p}^{-p} dx + k_1 \int_\Omega u^p f_{1p}^{1-p} dx.$$

Notice that in the previous inequality we have used the fact that

$$k_{0i} \leq \chi_i(w) exp\left\{\int_{\underline{w}}^w \chi_i(s) ds\right\} \leq \chi_i(w) f_{ip}(w).$$

We replace (3.12) in Lemma 3.4 and due to

$$1 \leq \frac{1}{1 - \varepsilon\frac{p-1}{p}} \leq \frac{1}{1-\varepsilon},$$

we achieve

$$\frac{1}{p-1}\frac{d}{dt}\int_\Omega u^p f_{1p}^{1-p} dx \leq -\frac{\varepsilon_u k_{01}}{1-\varepsilon\frac{p-1}{p}}\int_\Omega u^{p+1} f_{1p}^{-p} dx + \frac{k_1}{1-\varepsilon\frac{p-1}{p}}\int_\Omega u^p f_{1p}^{1-p} dx$$

$$\leq -\varepsilon_u k_{01}\int_\Omega u^{p+1} f_{1p}^{-p} dx + \frac{k_1}{1-\varepsilon}\int_\Omega u^p f_{1p}^{1-p} dx.$$

Repeating the process for v, the proof ends. □

LEMMA 3.6. *Under assumptions* (1.3)–(1.16), *the solutions to* (1.1)–(1.2) *satisfy*

$$\|u\|_{L^\infty(\Omega)} \leq \exp\left\{\frac{1}{1-\epsilon}\int_{\underline{w}}^{\overline{w}} \chi_1(s) ds\right\} \max\left\{k_1\left(\epsilon_u k_{01}(1-\varepsilon)\right)^{-1}, \|u_0\|_{L^\infty(\Omega)}\right\},$$

$$\|v\|_{L^\infty(\Omega)} \leq \exp\left\{\frac{1}{1-\epsilon}\int_{\underline{w}}^{\overline{w}} \chi_2(s) ds\right\} \max\left\{k_2\left(\epsilon_v k_{02}(1-\varepsilon)\right)^{-1}, \|v_0\|_{L^\infty(\Omega)}\right\}$$

for any $t < T^*$.

Proof. The proof of the lemma is based on an iterative method for the function

$$X_p = \int_\Omega u^p f_{1p}^{1-p}$$

with f_{1p} as in (3.6). Taking a positive constant $s > 0$ and splitting X_p into two different integrals depending on s, it holds that

$$\int_\Omega u^p f_{1p}^{1-p} dx = \int_{f_{1p} \leq \epsilon_u k_{01} su} u^p f_{1p}^{1-p} dx + \int_{f_{1p} > \epsilon_u k_{01} su} u^p f_{1p}^{1-p} dx$$

$$\leq s\epsilon_u k_{01} \int_{f_{1p} \leq \epsilon_u k_{01} su} u^{p+1} f_{1p}^{-p} dx + (\epsilon_u k_{01} s)^{1-p} \int_{f_{1p} \geq \epsilon_u k_{01} su} u dx.$$

Since

$$\int_{f_{1p} \leq \epsilon_u k_{01} su} u^{p+1} f_{1p}^{-p} dx \leq \int_\Omega u^{p+1} f_{1p}^{-p} dx \quad \text{and} \quad \int_{f_{1p} \geq \epsilon_u k_{01} su} u dx \leq \int_\Omega u dx,$$

we have

$$(3.13) \qquad -\epsilon_u k_{01} \int_\Omega u^{p+1} f_{1p}^{-p} dx \leq -\frac{1}{s} \int_\Omega u^p f_{1p}^{1-p} dx + s^{-p} \epsilon_u^{1-p} k_{01}^{1-p} \int_\Omega u dx.$$

By Lemma 3.5 and thanks to (3.13), it gives

$$(3.14) \qquad \frac{1}{p-1} \frac{d}{dt} X_p \leq \left(\frac{k_1}{1-\varepsilon} - \frac{1}{s} \right) X_p + s^{-p} \epsilon_u^{1-p} k_{01}^{1-p} \int_\Omega u dx.$$

We take $s^{-1} > \frac{k_1}{1-\varepsilon}$ and apply the maximum principle to the ODE to obtain a global bound for X_p,

$$X_p^{\frac{1}{p}} \leq \max\{k_1 \left(\epsilon_u k_{01}(1-\varepsilon) \right)^{-1}, X_p^{\frac{1}{p}}(0)\};$$

taking limits when $p \to \infty$ we have that $f_{1p} \to f_{1\infty}$; as in (1.16),

$$f_{1\infty}(\overline{w}) = exp \left\{ \frac{1}{1-\epsilon} \int_{\underline{w}}^{\overline{w}} \chi_1(s) ds \right\},$$

and then

$$\lim_{p \to \infty} X_p^{\frac{1}{p}} \leq \max\{k_1 \left(\epsilon_u k_{01}(1-\varepsilon) \right)^{-1}, X_p^{\frac{1}{p}}(0)\}.$$

Therefore, it gives

$$\|u\|_{L^\infty(\Omega)} \leq f_{1\infty}(\overline{w}) \max \left\{ k_1 \left(\epsilon_u k_{01}(1-\varepsilon) \right)^{-1}, \|u_0\|_{L^\infty(\Omega)} \right\}.$$

The proof for v is done in the same way and we omit the details. $\square$

To end the proof of the theorem, we introduce the auxiliary problem where the function h in (1.1) is replaced by the truncation

$$\tilde{h}(u,v,w) = \begin{cases} h(u,v,w) & \text{if } w \leq \overline{w} \\ h(u,v,\overline{w}) & \text{if } w > \overline{w}. \end{cases}$$

We introduce the unknowns $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ defined as the solutions of the system

$$(3.15) \qquad \begin{cases} \tilde{u}_t = \Delta\tilde{u} - \nabla \cdot (\tilde{u}\chi_1(\tilde{w})\nabla\tilde{w}), & x \in \Omega, \ t > 0, \\ \tilde{v}_t = \Delta\tilde{v} - \nabla \cdot (\tilde{v}\chi_2(\tilde{w})\nabla\tilde{w}), & x \in \Omega, \ t > 0, \\ \tilde{w}_t = \varepsilon\Delta\tilde{w} + \tilde{h}(\tilde{u},\tilde{v},\tilde{w}), & x \in \Omega, \ t > 0, \\ \nabla\tilde{u} \cdot \nu = \nabla\tilde{v} \cdot \nu = \nabla\tilde{w} \cdot \nu = 0, & x \in \partial\Omega, \ t > 0 \end{cases}$$

for $\varepsilon > 0$, and initial data

$$\tilde{u}(x,0) = u_0(x), \quad \tilde{v}(x,0) = v_0(x), \quad \tilde{w}(0,x) = w_0(x), \qquad x \in \Omega.$$

As in Lemmas 3.2–3.6 we have that the solution exists in an interval $(0, \tilde{T}_{\max})$ for $\tilde{T}_{\max}$ defined in the same fashion as (3.5). Then

$$\|\tilde{u}\|_{L^\infty(\Omega)} \le f_{1\infty}(\overline{w}) \max \left\{ k_1 \left(\epsilon_u k_{01}(1-\varepsilon) \right)^{-1}, \|u_0\|_{L^\infty(\Omega)} \right\},$$

$$\|\tilde{v}\|_{L^\infty(\Omega)} \le f_{2\infty}(\overline{w}) \max \left\{ k_2 \left(\epsilon_u k_{02}(1-\varepsilon) \right)^{-1}, \|v_0\|_{L^\infty(\Omega)} \right\}$$

as far as $\tilde{w} \le \overline{w}$. We define $\tilde{T}^*$ by analogy with T^* in (3.11), and prove that $\tilde{T}^* = \tilde{T}_{\max}$ by contradiction: Assume that

(3.16) $$\tilde{T}^* < \tilde{T}_{\max}$$

and apply the maximum principle to

$$\tilde{w}_t = \varepsilon \Delta \tilde{w} + \tilde{h}(\tilde{u}, \tilde{v}, \tilde{w}) \le h(\overline{u}, \overline{v}, w) \qquad (0, T^*),$$

to obtain, thanks to (1.7) and (1.13), that $\tilde{w} < \overline{w}$. Which contradicts (3.16) and proves $\tilde{T}^* = \tilde{T}_{\max}$ and then $\tilde{T}_{\max} = \infty$.

Notice that, since $\tilde{w} \le \overline{w}$ we have that $(\tilde{u}, \tilde{v}, \tilde{w})$ is also a solution to (1.1) and we have that

$$T_{\max} \ge \tilde{T}_{\max} = \infty$$

which ends the proof of the theorem.

4. Asymptotic behavior. In this section we survey the asymptotic behavior of the solutions of the dissipative system for $0 < \varepsilon < 1$. We formulate a theorem and simplify the system to the case where h is a linear function given by

(4.1) $$h(u,v,w) = au + v - 2\mu w \quad (\mu, a > 0).$$

Denoting by $C(\Omega)$ the smallest positive constant such that

$$\int_\Omega u^2 dx \le C(\Omega) \int_\Omega |\nabla u|^2 dx$$

for all functions u in $H^1(\Omega)$ satisfying $\int_\Omega u(x)dx = 0$, we have that, $C(\Omega) = C(\Omega_0)\rho^2$ if $\Omega = \rho\Omega_0$ (Ω_0 a fixed domain and $0 < \rho < 1$).

To prove the main result of this section we add the following necessary conditions:

(4.2) $$C(\Omega) \max \left\{ (\chi_1(\underline{w})\overline{u})^2, \ (\chi_2(\underline{w})\overline{v})^2 \right\} \max\{1, a^2\} < 2\mu\varepsilon.$$

THEOREM 4.1. *For any initial data $(u_0(x), v_0(x), w_0(x))$ satisfying (4.2), the unique global solution (u, v, w) of system (1.1) has the asymptotic behavior*

$$\int_\Omega |u - u^*|^2 dx \to 0, \quad \int_\Omega |v - v^*|^2 dx \to 0, \quad \int_\Omega |w - w^*|^2 dx \to 0 \quad as \quad t \to \infty,$$

where (u^*, v^*, w^*) is defined in (1.8) and $w^* = (au^* + v^*)/2\mu$.

Proof. Integrating the last equation of (1.1) over Ω we get

$$\int_\Omega w_t dx + 2\mu \int_\Omega w dx = au^* + v^*.$$

Using the notation $W(t) := \int_\Omega w(x,t)dx$, we achieve that the solution W of the above first order linear differential equation satisfies

(4.3)
$$W(t) = \frac{au^* + v^*}{2\mu} + c_0 e^{-2\mu t} \quad \left(c_0 = W(0) - \frac{au^* + v^*}{2\mu} \right).$$

Moreover, by (4.1) and (4.3), we have

$$
\begin{aligned}
\int_\Omega h(u,v,w)w dx &= \int_\Omega (au + v - 2\mu w)w dx \\
&= \int_\Omega [a(u-u^*) + (v-v^*) - 2\mu(w-W)](w-W)dx \\
&\quad + W(au^* + v^* - 2\mu W) \\
&= \int_\Omega [a(u-u^*) + (v-v^*)](w-W)dx \\
&\quad - 2\int_\Omega \mu(w-W)^2 dx + O(e^{-2\mu t}).
\end{aligned}
$$

(4.4)

Applying Schwarz's inequality,

$$\int_\Omega [a(u-u^*)+(v-v^*)](w-W)dx \le 2\mu \int_\Omega (w-W)^2 dx + \frac{1}{4\mu}\int_\Omega [a^2(u-u^*)^2 + (v-v^*)^2]dx$$

and substituting it into (4.4), by Poincaré's inequality we obtain

(4.5)
$$\int_\Omega h(u,v,w)w dx \le \frac{C(\Omega)}{4\mu}\int_\Omega [a^2 \mid \nabla u \mid^2 + \mid \nabla v \mid^2]dx + O(e^{-2\mu t}).$$

Multiplying by u and by v the first two equations in (1.1) and integrating over Ω_T, we have

(4.6)
$$\frac{1}{2}\int_\Omega u^2\vert_0^T dx + \int\int_{\Omega_T} \mid \nabla u \mid^2 dxdt = \int\int_{\Omega_T} u\chi_1(w)\nabla u \cdot \nabla w dxdt,$$

(4.7)
$$\frac{1}{2}\int_\Omega v^2\vert_0^T dx + \int\int_{\Omega_T} \mid \nabla v \mid^2 dxdt = \int\int_{\Omega_T} v\chi_2(w)\nabla v \cdot \nabla w dxdt.$$

Taking the scalar product with w and integrating the last equation in (1.1) over Ω_T, adding it to (4.6) and (4.7) after a multiplication by a positive constant λ, we have

(4.8)
$$
\begin{aligned}
&\frac{\lambda}{2}\int_\Omega [u^2 + v^2]\vert_0^T dx + \lambda\int\int_{\Omega_T} (\mid \nabla u \mid^2 + \mid \nabla v \mid^2)\, dxdt \\
&\quad + \frac{1}{2}\int_\Omega w^2\vert_0^T + \varepsilon\int\int_{\Omega_T} \mid \nabla w \mid^2 dxdt \\
&= \lambda\int\int_{\Omega_T} (u\chi_1(w)\nabla u + v\chi_2\nabla v)\,\nabla w dxdt + \int\int_{\Omega_T} h(u,v,w)w dxdt.
\end{aligned}
$$

The monotony of χ_i given by $\chi_i'(w) < 0$ implies $u\chi_1 \leq \overline{u}\chi_1(\underline{w}) := \alpha$ and $v\chi_2 \leq \overline{v}\chi_2(\underline{w}) := \beta$. Applying Schwarz's inequality to the first integral on the right-hand side in (4.8), we get

$$(4.9) \quad \lambda \int \int_{\Omega_T} u\chi_1 \nabla u \cdot \nabla w \, dx dt \leq \frac{\varepsilon}{2} \int \int_{\Omega_T} |\nabla w|^2 \, dx dt + \frac{\lambda^2 \alpha^2}{2\varepsilon} \int \int_{\Omega_T} |\nabla u|^2 \, dx dt$$

and

$$(4.10) \quad \lambda \int \int_{\Omega_T} v\chi_2 \nabla v \cdot \nabla w \, dx dt \leq \frac{\varepsilon}{2} \int \int_{\Omega_T} |\nabla w|^2 \, dx dt + \frac{\lambda^2 \beta^2}{2\varepsilon} \int \int_{\Omega_T} |\nabla v|^2 \, dx dt.$$

Substituting (4.5), (4.9), and (4.10) into (4.8) we deduce the bounds

$$\left(\lambda - \lambda^2 \left(\frac{\alpha^2}{2\varepsilon}\right) - \frac{a^2 C(\Omega)}{4\mu}\right) \int \int_{\Omega_T} |\nabla u|^2 dx dt \leq C,$$

$$\left(\lambda - \lambda^2 \left(\frac{\beta^2}{2\varepsilon}\right) - \frac{C(\Omega)}{4\mu}\right) \int \int_{\Omega_T} |\nabla v|^2 dx dt \leq C.$$

We denote by $\gamma = \max\{\alpha, \beta\}$ and we prove that there exists a positive constant λ such that

$$(4.11) \quad \left(\lambda - \lambda^2 \left(\frac{\gamma^2}{2\varepsilon}\right) - \frac{C(\Omega)}{4\mu} \max\{1, a^2\}\right) \int \int_{\Omega_T} [|\nabla u|^2 + |\nabla v|^2] dx dt \leq C.$$

For this purpose, under condition (4.2), i.e.,

$$C(\Omega)\gamma^2 \max\{1, a^2\} < 2\mu\varepsilon,$$

we demonstrate that the quadratic equation in λ has two positive roots, $0 < \lambda_1 < \lambda_2$, and by choosing any $\lambda \in (\lambda_1, \lambda_2) \neq \emptyset$, we obtain

$$\lambda - \lambda^2 \left(\frac{\gamma^2}{2\varepsilon}\right) - \frac{C(\Omega)}{4\mu} \max\{1, a^2\} > 0,$$

hence, (4.11) is reduced to

$$\int \int_{\Omega_T} [|\nabla u|^2 + |\nabla v|^2] dx dt \leq C.$$

By (4.5) and (4.8) we derive the same bound in ∇w, than

$$\int \int_{\Omega_\infty} |\nabla u|^2 dx dt + \int \int_{\Omega_\infty} |\nabla v|^2 dx dt + \int \int_{\Omega_\infty} |\nabla w|^2 dx dt \leq C.$$

To finish the proof, we follow the steps of Lemma 3.4 in [18] with $\mu_1 = \mu_2 = 0$. Thereby we define

$$k(t) := \int_\Omega \left[(u(x, t) - u^*)^2 + (v(x, t) - v^*)^2\right] dx.$$

Thanks to (4.14) and Poincare's inequality,

$$(4.12) \quad \int_0^\infty k(t) dt \leq \int_{\Omega_\infty} \left(|\nabla u|^2 + |\nabla v|^2\right) dx dt \leq C.$$

In order to have the limit $k(t) \to 0$, as $t \to \infty$, we apply Lemma 5.1 (ii) of [11] and we need to prove that

(4.13) $\qquad |k(t+s) - k(t)| \leq \epsilon(t) \quad \text{for all} \quad s > 0, \quad \text{where } \epsilon(t) \to 0 \quad \text{as} \quad t \to \infty.$

Notice that

$$\int_\Omega [(u(x,t+s) - u^*)^2 - (u(x,t) - u^*)^2] dx$$
$$= \int_\Omega [u^2(x,t+s) + (u^*)^2 - 2u^* u(x,t+s) - u^2(x,t) - (u^*)^2 + 2u^* u(x,t)] dx$$

and

$$\int_\Omega u^* u(x,t+s) dx = \int_\Omega u^* u(x,t) dx = (u^*)^2$$

and therefore

$$\int_\Omega [(u(x,t+s) - u^*)^2 - (u(x,t) - u^*)^2] dx = \int_\Omega [u^2(x,t+s) - u^2(x,t)] dx.$$

Since

(4.14) $$k(t+s) - k(t) = \int_t^{t+s} k'(\tau) d\tau$$

and

$$k'(t) = 2\int_\Omega (u - u^*) u_t dx + 2\int_\Omega (v - v^*) v_t dx = 2\int_\Omega u u_t dx + 2\int_\Omega v v_t dx,$$

multiplying the first equation in (1.1) by u we have that

$$\int_\Omega u u_t dx = \int_\Omega |\nabla u|^2 dx + \int_\Omega \chi_1(w) u \nabla u \nabla w dx.$$

By the inequalities

$$\int_\Omega \chi_1(w) u \nabla u \nabla w dx \leq \|u\|_{L^\infty(\Omega_\infty)} \chi_1(\underline{w}) \int_\Omega \left[|\nabla u|^2 + |\nabla w|^2\right] dx,$$
$$\int_\Omega \chi_2(w) v \nabla v \nabla w dx \leq \|v\|_{L^\infty(\Omega_\infty)} \chi_2(\underline{w}) \int_\Omega \left[|\nabla v|^2 + |\nabla w|^2\right] dx,$$

we obtain

(4.15) $$|k'(t)| \leq C \int_\Omega \left[|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\right] dx$$

and therefore, thanks to (4.14) and (4.15), we get

$$|k(t+s) - k(t)| \leq C \int_t^{t+s} \int_\Omega (|\nabla u(x,\tau)|^2 + |\nabla v(x,\tau)|^2 + |\nabla w(x,\tau)|^2) dx d\tau.$$

Therefore $\epsilon(t) \to 0$ as $t \to \infty$. We now apply Lemma 5.1 (ii) in [11] to obtain $k(t) \to 0$ as $t \to \infty$.

For the limit

$$\int_\Omega |w - w^*|^2 dx \to 0 \qquad \text{as } t \to \infty,$$

consider W defined in (4.3) and define the function

$$q(t) = \int_\Omega (w(x,t) - W(t))^2 dx.$$

To obtain $q(t) \to 0$ as $t \to \infty$, we recall again Lemma 5.1 (ii) in [11]. We have to prove that

$$\int_0^\infty q(t)dt < \infty \quad \text{and} \quad |q(t+s) - q(t)| \leq \epsilon(t) \to 0 \text{ as } t \to \infty.$$

To obtain the first inequality we consider the equation

$$w_t - W_t - \varepsilon\Delta(w - W) + 2\mu(w - W) = a(u - u^*) + v - v^*.$$

Multiplying by $w - W$, integrating in time, and thanks to (4.12) we obtain

$$\int_0^\infty q(t)dt < C.$$

The second inequality is proven in the same way as u and v. Thereby, we infer that

$$\int_\Omega |w(x,t) - W(t)|^2 dx \to 0 \text{ as } t \to \infty,$$

and thanks to (4.3) the proof ends.   □

Remark 4.1. Since any stationary solution (u^*, v^*, w^*) of (1.1), with $\underline{w} < w^*(x) < \overline{w}$, $0 < u^*(x) < \overline{u}$, $0 < v^*(x) < \overline{v}$ satisfies the estimate

$$\int\int_{\Omega_\infty} |\nabla u|^2 dxdt + \int\int_{\Omega_\infty} |\nabla v|^2 dxdt + \int\int_{\Omega_\infty} |\nabla w|^2 dxdt \leq C,$$

it follows that such solutions are necessarily constant.

5. Applications. In this section we apply the theoretical results obtained in the previous section to the case where the chemotactic sensitivities of the species χ_i are defined by

$$\chi_i = \alpha_i/(\beta_i + w) \quad \text{and} \quad h = au + v - 2\mu w,$$

with positive constants a, μ, α_i, and β_i (for $i = 1,2$) such that

(5.1) $\alpha_i \leq 1 - \varepsilon \qquad$ for $i = 1, 2,$

(5.2) $\beta_i < \dfrac{\alpha_i}{\alpha_i + 1} \qquad$ for $i = 1, 2.$

With α_i as in (5.1), the chemotactic sensitivities χ_i satisfy (1.4) for every $0 < \varepsilon < 1$ with $i = 1, 2$.

In order to obtain the global existence of the solutions of (1.1) and to prove that any solution is asymptotically stable, we have to verify that assumptions (1.6), (1.17), (1.11)–(1.13), and (4.2) are fulfilled.

1. We have $h_u = a > 0$, $h_v = 1$, and $h_w = -2\mu < 0$, so assumptions (1.6) and (1.7) are satisfied for $\epsilon_u = a$, $\epsilon_v = 1$, and $\epsilon_w = 2\mu$.
2. Relation (1.11) is equivalent to

$$(5.3) \qquad\qquad 2\mu\alpha_i \frac{w}{\beta_i + w} \le k_i.$$

For

$$k_i := 2\mu\alpha_i \frac{\overline{w}}{\beta_i + \overline{w}},$$

(5.3) holds, where the upper bound $\overline{w}$ is to be defined latter. Moreover, observe that it is enough to consider $k_i = 2\mu\alpha_i$, $i = 1, 2$.
3. Computing in (1.12), we take positive constants such that

$$k_{0i} := \frac{\alpha_i}{(\beta_i + \overline{w})^{1-\alpha_i}}, \qquad i = 1, 2.$$

4. Notice that $h(0,0,0) = 0$ and

$$h(\overline{u}, \overline{v}, \overline{w}) = a\overline{u} + \overline{v} - 2\mu\overline{w}.$$

Looking upon the lower bound $\underline{w} = 0$, for any upper bound $\overline{w}$, by expressions (1.14)–(1.16), the second inequality in (1.13) is equivalent to

$$(5.4) \qquad\qquad (\beta_1 + \overline{w})^{\frac{\alpha_1\varepsilon}{1-\varepsilon}} + (\beta_2 + \overline{w})^{\frac{\alpha_2\varepsilon}{1-\varepsilon}} < 1 - \varepsilon.$$

For every $\overline{w}$ and β_i verifying (5.4) with α_i as in (5.1), we have $h(\overline{u}, \overline{v}, \overline{w}) < 0$.
5. It remains to be studied what conditions are necessary to fulfill (1.17). Simplifying and operating, we found that (1.17) is reduced to

$$(5.5) \qquad (\beta_1 + \overline{w})^{\frac{\alpha_1\varepsilon}{1-\varepsilon}} \overline{w} \frac{\alpha_1}{\beta_1 + w} + (\beta_2 + \overline{w})^{\frac{\alpha_2\varepsilon}{1-\varepsilon}} \overline{w} \frac{\alpha_2}{\beta_2 + w} \le 1 - \varepsilon,$$

and looking back to (5.4), if we request

$$(5.6) \qquad\qquad \overline{w} < \frac{\beta_i}{\alpha_i}, \qquad i = 1, 2,$$

then (5.5) is verified.
For the stability, assumption (4.2) is reduced to

$$(5.7) \qquad C(\Omega) \left(\frac{\alpha_i}{\beta_i}\right) (\beta_i + \overline{w})^{\frac{2\alpha_i\varepsilon}{1-\varepsilon}} \max\{1, a^2\} < 2\mu\varepsilon(1-\varepsilon)^2.$$

Therefore, for every $\overline{w}$ as in (5.5) and (5.6), for all initial data (u_0, v_0, w_0) of (1.1) satisfying

$$||u_0||_{L^\infty} \le \frac{2\mu\overline{w}}{a(\beta_1 + \overline{w})^{2-\varepsilon}(1-\varepsilon)}, \qquad ||v_0||_{L^\infty} \le \frac{2\mu\overline{w}}{(\beta_2 + \overline{w})^{2-\varepsilon}(1-\varepsilon)}, \text{ and } 0 \le w_0 \le \overline{w}$$

such that

$$u_0 \not\equiv 0 \quad \text{and} \quad v_0 \not\equiv 0,$$

all the required hypotheses are verified and we can apply Theorems 3.1 and 4.1, i.e., the solution (u, v, w) of (1.1) is globally uniformly bounded and

$$\lim_{t\to\infty} \int_\Omega |u - u^*|^2 dx \;=\; \lim_{t\to\infty} \int_\Omega |v - v^*|^2 dx \;=\; \lim_{t\to\infty} \int_\Omega |w - w^*|^2 dx = 0,$$

where the stationary solution of system (1.1) is

$$(u^*, v^*, w^*) := \left(\int_\Omega u_0(x)dx, \int_\Omega v_0(x)dx, \int_\Omega \frac{(au_0 + v_0)dx}{2\mu} \right).$$

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