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## Asymptotic stability of a two species chemotaxis system with non-diffusive chemoattractant <sup>☆</sup>

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### Abstract

We study the behavior of two biological populations “ $u$ ” and “ $v$ ” attracted by the same chemical substance whose behavior is described in terms of second order parabolic equations. The model considers a logistic growth of the species and the interactions between them are relegated to the chemoattractant production. The system is completed with a third equation modeling the evolution of chemical. We assume that the chemical “ $w$ ” is a non-diffusive substance and satisfies an ODE, more precisely,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u), & x \in \Omega, t > 0, \\ v_t = \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - v), & x \in \Omega, t > 0, \\ w_t = h(u, v, w), & x \in \Omega, t > 0, \end{cases}$$

under appropriate boundary and initial conditions in an  $n$ -dimensional open and bounded domain  $\Omega$ . We consider the cases of positive chemo-sensitivities, not necessarily constant elements. The chemical production function  $h$  increases as the concentration of the species “ $u$ ” and “ $v$ ” increases. We first study the global existence and uniform boundedness of the solutions by using an iterative approach. The asymptotic stability of the homogeneous steady state is a consequence of the growth of  $h$ ,  $\chi_i$  and the size of  $\mu_i$ . Finally, some examples of the theoretical results are presented for particular functions  $h$  and  $\chi_i$ .

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## 1. Introduction

*Chemotaxis* is the ability of living organisms to orientate their movement towards or away from a chemical substance. The term was introduced to describe cell migration observed during the early days of the development of microscopy in the nineteenth century. As the technology advanced, the action has emerged as a relevant process in many biological situations, as immune system response, embryo development, tumor growth, bacteria cluster formation, etc. One of the most studied biological systems where chemotaxis occurs is the slime mold aggregation, where *Dictyostelium Discoideum* aggregates to concentrate the mass in a small region. In the last 40 years, after the pioneering works of Keller–Segel, chemotaxis has been described by using nonlinear systems of PDEs with second order terms modeling the aggregation of the organisms. The problem contains a set of parameters giving different weights to those terms which describe other biological processes involved as aggregation, diffusion, degradation, production, etc.

Chemotactic models appear not only in the mentioned biological processes at microscopic scale, but also population dynamics at macroscopic scale in the context of life sciences, *gravitational collapse* in astrophysics, material sciences, etc.

From a theoretical point of view, the problem presents important mathematical challenges, some of them already studied for systems of one species with one chemoattractant but still unclear for two species or multiple stimuli models. Some of these challenges are to describe the mechanism which drives the system to finite time blow-up or to global boundedness and to obtain the constrains of the parameters, the threshold values which decide the behavior and the stability of the system.

In that direction many authors have studied the qualitative properties of these mathematical models depending on the relations between such parameters and the initial data. The first mathematical works deal with the initial mass threshold in a two dimensional domain to obtain finite time blow-up. The fully Parabolic problem, i.e., where the species and the stimuli behavior are described by two parabolic equations and the Parabolic–Elliptic system where the stimuli satisfy a second order elliptic equation, have been studied in a deep way in different works, see for instance Horstmann [11] and references therein for more details. Subsequently, a Parabolic–ODE model has been derived by Stevens [17] (see also Levine and Sleeman [14] and Othmer and Stevens [16]) to modelize the aggregation of myxobacteria by using a discrete model and probabilities to describe the oriented movement of the organisms. The stability of the Parabolic–ODE problem has been studied in several works, in Friedman and Tello [10] the local stability is obtained under assumptions

$$\frac{\partial h}{\partial u} > 0 \quad \text{and} \quad u\chi \frac{\partial h}{\partial u} + \frac{\partial h}{\partial w} < 0.$$

Motivated by biological experiments, see Lauffenburger [13], multi-species chemotaxis systems become a rich mathematical problem, already proposed in the 1980s by Alt [1] and subsequently studied in Fasano, Mancini and Primicerio [9], Wolansky [21] and Horstmann [12], among others. More recently, systems of two species with one chemoattractant have been studied by different research groups, the finite-time blow-up in bounded domains for the Parabolic–Parabolic–Elliptic issue has been analyzed by Espejo, Stevens and Velázquez [7] and [8] for simultaneous and non-simultaneous blow-up. See also the results in Biler, Espejo and Guerra [3], Biler and Guerra [4] for bounded domains and Conca and Espejo [5,6] for the two-dimensional case in the whole space. The Parabolic–Parabolic–Elliptic cases with competitive terms, i.e.,

when there exists an explicit interaction between the species, have also been studied in the last years by different authors. In Tello and Winkler [20] the Parabolic–Parabolic–Elliptic problem is analyzed under “weak” competitive assumptions, which drive it to positive homogeneous steady states. A more general case is considered in Negreanu and Tello [18] where both species persist in time. “Strong competition assumptions”, i.e., when the competitive parameters between the species are “large”, drive the system to the extinction of the weaker species, see Stinner, Tello and Winkler [22] for details. Parabolic–Parabolic–Elliptic models with external application of chemoattractant have also been considered in Negreanu and Tello [19] where the stability of the solutions is described for a large range of parameters.

In the present work we analyze the Parabolic–Parabolic–ODE system where the interaction of the species is relegated to the chemoattractant production, i.e., there is no competition/cooperation or symbiosis between the species. The chemotactic sensitivities of the species,  $\chi_1$  and  $\chi_2$ , are not necessary constant, but they only depend on the chemoattractant. Denoting the population densities by  $u(x, t)$  and  $v(x, t)$  and the concentration of the chemoattractant by  $w(x, t)$ , classical models lead to

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1 - u), & x \in \Omega, t > 0, \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1 - v), & x \in \Omega, t > 0, \\ w_t = h(u, v, w), & x \in \Omega, t > 0, \end{cases} \quad (1.1)$$

in a bounded and regular domain  $\Omega \subset \mathbb{R}^n$ , for  $n \geq 1$ , with Neumann boundary conditions

$$\nabla u \cdot \nu - u\chi_1(w)\nabla w \cdot \nu = \nabla v \cdot \nu - v\chi_2(w)\nabla w \cdot \nu = 0, \quad x \in \partial\Omega, t > 0, \quad (1.2)$$

and bounded initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(0, x) = w(x), \quad x \in \Omega, \quad (1.3)$$

in  $(W^{1,s}(\Omega))^3$ , with  $s \in [\max\{4, N\}, \infty)$ , and positive parameters  $\mu_1$  and  $\mu_2$ .

Notice that  $h(u, v, w)$  represents the balance between the production of the chemical substance by the living organisms and its natural degradation. Depending on the process, the chemotactic functions  $\chi_1$  and  $\chi_2$  can take different forms, the simplest case, where  $\chi_i$  are constants is treated in several examples in the present work. We assume along this work that  $\chi_i$  (for  $i = 1, 2$ ) and  $h$  are regular and the chemoattractant sensitivities are positive, i.e.

$$\chi_i, h \in W_{loc}^{1,\infty}(\mathbb{R}_+^2 \times \mathbb{R}), \quad \chi_i > 0. \quad (1.4)$$

We consider the case where the production of the chemical increases as the concentration of the species increases, i.e.,

$$\frac{\partial h}{\partial u} \geq \epsilon_u > 0 \quad \text{and} \quad \frac{\partial h}{\partial v} \geq \epsilon_v > 0. \quad (1.5)$$

Taking into account the natural degradation of  $w$ , we set

$$\frac{\partial h}{\partial w} < 0. \quad (1.6)$$

By Implicit Function Theorem and assumption (1.6) we deduce the existence of a unique constant  $\tilde{w}$  satisfying

$$h(1, 1, \tilde{w}) = 0. \tag{1.7}$$

Consequently  $(1, 1, \tilde{w})$  is a constant stationary solution of the system.

In Sections 2 and 3 we consider the Parabolic–Parabolic–ODE problem, where the chemosensitivity functions satisfy

$$-h(0, 0, w) \leq \frac{k_i}{\chi_i(w)} \quad \text{for some } k_i > 0, \tag{1.8}$$

$$0 < k_{0i} \leq \chi_i(w) e^{\int_{\underline{w}}^w \chi_i(s) ds} \quad \text{for } w > \underline{w}, \tag{1.9}$$

for some positive constants  $k_{0i}$ , with  $i = 1, 2$ . We assume that there exist positive constants  $\bar{w}$  and  $\underline{w}$  such that

$$\underline{w} < \tilde{w} < \bar{w},$$

the chemical production  $h$  fulfills

$$h(u, v, \underline{w}) \geq 0, \quad h(u, v, \bar{w}) \leq 0, \quad \text{for } 0 \leq u \leq \bar{u}, \quad 0 \leq v \leq \bar{v}, \tag{1.10}$$

where

$$\bar{u} := f_1(\bar{w}) \max\{(k_1 + \mu_1)(\epsilon_u k_{01} + \mu_1)^{-1}, \|u_0\|_{L^\infty(\Omega)}\}, \tag{1.11}$$

$$\bar{v} := f_2(\bar{w}) \max\{(k_2 + \mu_2)(\epsilon_v k_{02} + \mu_2)^{-1}, \|v_0\|_{L^\infty(\Omega)}\}, \tag{1.12}$$

for  $f_i$  defined by

$$f_i(w) = e^{\int_{\underline{w}}^w \chi_i(s) ds}, \quad i = 1, 2 \tag{1.13}$$

and the initial data  $u_0 \neq 0$ ,  $v_0 \neq 0$  and  $w_0$  satisfy

$$0 \leq u_0 \in L^\infty(\Omega), \quad 0 \leq v_0 \in L^\infty(\Omega), \quad \underline{w} \leq w_0 \leq \bar{w}. \tag{1.14}$$

Section 2 focuses on the global existence of solutions by using an iterative method to prove uniform boundedness of the solutions in  $L^\infty(\Omega)$ . Section 3 is devoted to the stability of the homogeneous steady states. Using energy estimates, we get that the steady state  $(1, 1, \tilde{w})$  is asymptotically stable and any solution with initial data satisfying (1.14) converges to the constant steady state  $(1, 1, \tilde{w})$ . The result is obtained under the following constrain:

There exists  $\alpha \in (0, 1)$  such that

$$\alpha h_w + u h_u \chi_1 + v h_v \chi_2 < 0 \quad \text{and} \quad 2\sqrt{1 - \alpha} h_w + u h_v \chi_1 + v h_u \chi_2 < 0 \tag{1.15}$$

for any  $(u, v, w)$  satisfying

$$0 \leq u \leq \bar{u}, \quad 0 \leq v \leq \bar{v}, \quad \underline{w} \leq w \leq \bar{w}.$$

Notice that (1.15) is also satisfied by the initial data, since  $u_0 \leq \bar{u}$ ,  $v_0 \leq \bar{v}$  and  $w_0$  satisfies (1.14). We also prove that replacing (1.15) by

$$2u\chi_1 \frac{\partial h}{\partial u} + \frac{\partial h}{\partial w} < 0, \quad 2v\chi_2 \frac{\partial h}{\partial v} + \frac{\partial h}{\partial w} < 0, \quad \text{if } 0 \leq u \leq \bar{u}, \quad 0 \leq v \leq \bar{v}, \quad \underline{w} \leq w \leq \bar{w}, \quad (1.16)$$

$$\left| \frac{uh_u\chi_1 + h_w}{h_u^2} - \frac{vh_v\chi_2 + h_w}{h_v^2} \right| \leq \frac{\sqrt{h_w(2uh_u\chi_1 + h_w)}}{h_u^2} + \frac{\sqrt{h_w(2vh_v\chi_2 + h_w)}}{h_v^2}, \quad (1.17)$$

the result of asymptotic behavior remains valid.

All these hypotheses indicate us how to choose the initial data  $(u_0, v_0, w_0)$  of the problem to obtain the desired results: global existence, global boundedness and asymptotic stability, i.e., the last ones are required in the study of the stability of the homogeneous steady states.

From a biological point of view, these restrictions show a balance between degradation and production, i.e., the degradation is, in some sense, “stronger” than the production of the chemical and it is a sufficient condition for the global boundedness.

**Remark 1.1.**

1. It should be emphasized that for the case  $h_u = h_v$ , expression (1.17) does not impose any further restrictions on the choice of the initial data  $(u_0, v_0)$ , i.e., it could be simplified as follows

$$\begin{aligned} \left| \frac{uh_u\chi_1 + h_w}{h_u^2} - \frac{vh_v\chi_2 + h_w}{h_v^2} \right| &= \left| \frac{1}{2h_u^2 h_w} [h_w(h_w + 2uh_u\chi_1) - h_w(h_w + 2vh_u\chi_2)] \right| \\ &\leq \frac{h_w(h_w + 2uh_u\chi_1)}{2h_u^2 |h_w|} + \frac{h_w(h_w + 2vh_u\chi_2)}{2h_u^2 |h_w|}. \end{aligned}$$

By subsequent, for the case  $h_u = h_v > 0$ , (1.17) is reduced to ensure that

$$\begin{aligned} &h_w(h_w + 2uh_u\chi_1) + h_w(h_w + 2vh_u\chi_2) \\ &\leq 2|h_w|\sqrt{h_w(2uh_u\chi_1 + h_w)} + 2|h_w|\sqrt{h_w(2vh_u\chi_2 + h_w)}. \end{aligned}$$

The last inequality, taking term by term, i.e.,

$$(\sqrt{h_w(h_w + 2uh_u\chi_1)})^2 \leq 2|h_w|\sqrt{h_w(2uh_u\chi_1 + h_w)}$$

and

$$(\sqrt{h_w(h_w + 2vh_u\chi_2)})^2 \leq 2|h_w|\sqrt{h_w(2uh_u\chi_1 + h_w)}$$

is immediate, recalling (1.16), (1.5) and (1.6).

2. The same remark can be made about (1.15) in the particular case  $h_u = h_v$ , i.e.,

$$uh_u\chi_1 + vh_u\chi_2 < \min\{\alpha, 2\sqrt{1-\alpha}\}h_w.$$

Taking  $\alpha = 2(\sqrt{2} - 1)$ , the initial data of the problem are selected under the unique standard

$$h_u \max_{x \in \Omega} (u_0(x)\chi_1 + v_0(x)\chi_2) < -2(\sqrt{2} - 1)h_w. \tag{1.18}$$

These observations are very important at the time of studying the asymptotic behavior for stationary solutions of system (1.1) because it facilitates the choice of the initial data and the necessary parameters for its resolution.

In Section 4, we shall see, in a practical way, the direct implementation of these simplifications with several examples to illustrate the theoretical results. All the examples are chosen among those appear in the literature. Throughout the article we use the following notations

$$\Omega_T = \Omega \times (0, T), \quad \Omega_\infty = \Omega \times (0, \infty).$$

## 2. Global existence of solutions

The main aim of this section is to demonstrate the global existence of solutions which is described by the following theorem.

**Theorem 2.1.** *Under assumptions (1.4)–(1.6) and (1.8)–(1.10) there exists a unique solution*

$$(u, v, w) \in C([0, \infty), (W^{1,s}(\Omega))^3) \cap C^1((0, \infty), ((W^{1,s}(\Omega))')^2 \times W^{1,s}(\Omega))$$

to the problem (1.1)–(1.3) for any initial data  $(u_0, v_0, w_0) \in (W^{1,s}(\Omega))^3$ , with  $s \in [\max\{4, N\}, \infty)$  satisfying (1.14). Moreover the solution is uniformly bounded, i.e.

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \leq C < \infty.$$

The proof is split into several steps. The following lemma guarantees the local existence of the solution to the problem.

**Lemma 2.2.** *Under assumptions of Theorem 2.1, there exist a  $T_{max} > 0$  and a unique maximal positive solution to (1.1)–(1.3) satisfying*

$$(u, v, w) \in C([0, T_{max}), (W^{1,s}(\Omega))^3) \cap C^1((0, T_{max}), ((W^{1,s}(\Omega))')^2 \times W^{1,s}(\Omega))$$

and

$$\lim_{t \rightarrow T_{max}} (\|u(t)\|_{W^{1,s}(\Omega)} + \|v(t)\|_{W^{1,s}(\Omega)} + \|w(t)\|_{W^{1,s}(\Omega)} + t) = \infty.$$

**Proof.** We denote by  $b_1$  the following vector

$$b_1 = (u\chi_1(w), v\chi_2(w)).$$

For  $\phi = (u, v)$ , defining

$$\mathcal{A}_1\phi = (-\Delta u, -\Delta v); \quad \mathcal{A}_2(\phi, w)w = (\operatorname{div}(u\chi_1(w)\nabla w), \operatorname{div}(v\chi_2(w)\nabla w))$$

and  $T_{max}$  is maximal in the sense that

$$\mathcal{B}_1\phi = \frac{\partial\phi}{\partial n}, \quad \mathcal{B}_2(\phi, w)w = b_1 \frac{\partial w}{\partial n},$$

system (1.1)–(1.2) can be written as follows

$$\begin{cases} \phi_t + \mathcal{A}_1\phi + \mathcal{A}_2(\phi, w)w = g(\phi, w) & \text{in } \Omega \times (0, T_{max}), \\ w_t = h(\phi, w) & \text{in } \Omega \times (0, T_{max}), \\ \mathcal{B}_1\phi + \mathcal{B}_2(\phi, w)w = 0 & \text{on } \partial\Omega \times (0, T_{max}), \end{cases}$$

where  $g(\phi, w) = (\mu_1 u(1-u), \mu_2 v(1-v))$ .

Applying Theorem 6.4 of [2] we get the existence of a maximal weak solution.  $\square$

**Lemma 2.3.** Under assumptions of Theorem 2.1, we have that

$$u \geq 0, \quad v \geq 0, \quad w \geq \underline{w}, \quad \text{for } t \in (0, T_{max}).$$

**Proof.** In order to obtain the non-negativity of  $u$  and  $v$ , we work with the change of variables  $\tilde{u}$  and  $\tilde{v}$  given by

$$u = f_1(w)\tilde{u}, \quad v = f_2(w)\tilde{v}, \quad (2.19)$$

where  $f_i$  are defined in (1.13) by

$$f_i(w) = e^{\int_{\underline{w}}^w \chi_i(s) ds}, \quad i = 1, 2.$$

Notice that

$$\begin{aligned} u_t &= f_1'(w)w_t\tilde{u} + f_1(w)\tilde{u}_t, & v_t &= f_2'(w)w_t\tilde{v} + f_2(w)\tilde{v}_t \\ \nabla u &= f_1'(w)\nabla w\tilde{u} + f_1(w)\nabla\tilde{u}, & \nabla v &= f_2'(w)\nabla w\tilde{v} + f_2(w)\nabla\tilde{v}. \end{aligned}$$

Taking into account the expression of  $f_i$  in (1.13), we have

$$f_i'(w) = f_i(w)\chi_i(w), \quad i = 1, 2$$

and system (1.1)–(1.3) becomes

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} + \chi_1(w) \nabla w \nabla \tilde{u} + g_1(\tilde{u}, \tilde{v}, w), & x \in \Omega, t > 0, \\ \tilde{v}_t = \Delta \tilde{v} + \chi_2(w) \nabla w \nabla \tilde{v} + g_2(\tilde{u}, \tilde{v}, w), & x \in \Omega, t > 0, \\ w_t = h(f_1(w)\tilde{u}, f_2(w)\tilde{v}, w), & x \in \Omega, t > 0, \end{cases} \quad (2.20)$$

with Neumann boundary conditions

$$\frac{\partial \tilde{u}}{\partial n} = \frac{\partial \tilde{v}}{\partial n} = 0 \quad (2.21)$$

and initial data

$$\begin{aligned} \tilde{u}(x, 0) := \tilde{u}_0(x) &= \frac{u_0(x)}{f_1(w_0(x))}, & \tilde{v}(x, 0) := \tilde{v}_0(x) &= \frac{v_0(x)}{f_2(w_0(x))}, \\ w(0, x) &= w_0(x), \end{aligned} \quad (2.22)$$

where

$$\begin{aligned} g_1(\tilde{u}, \tilde{v}, w) &= -\tilde{u} \chi_1(w) h(f_1(w)\tilde{u}, f_2(w)\tilde{v}, w) + \mu_1 \tilde{u} (1 - f_1(w)\tilde{u}) \\ g_2(\tilde{u}, \tilde{v}, w) &= -\tilde{v} \chi_2(w) h(f_1(w)\tilde{u}, f_2(w)\tilde{v}, w) + \mu_2 \tilde{v} (1 - f_2(w)\tilde{v}). \end{aligned}$$

Standard results of Maximum Principle for parabolic equations and regularity of  $\chi_i$  and  $h$  prove the positivity of  $u$  and  $v$  since  $g_1(0, \tilde{v}, w) = g_2(\tilde{u}, 0, w) = 0$ . Thanks to (1.6), (1.10) and by the Maximum Principle applied to the ODE  $w_t = h(u, v, w)$ , we obtain

$$\underline{w} < w,$$

which ends the proof.  $\square$

**Lemma 2.4.** *The solution to (1.1)–(1.3) satisfies*

$$\int_{\Omega} u dx \leq \max \left\{ |\Omega|, \int_{\Omega} u_0 dx \right\} \quad \text{and} \quad \int_{\Omega} v dx \leq \max \left\{ |\Omega|, \int_{\Omega} v_0 dx \right\}.$$

**Proof.** Integrating (1.1) over  $\Omega$  and using (1.2), we have that

$$\frac{d}{dt} \int_{\Omega} u dx = \mu_1 \left( \int_{\Omega} u dx - \int_{\Omega} u^2 dx \right).$$

Thanks to Cauchy–Schwarz inequality

$$\frac{1}{|\Omega|} \left| \int_{\Omega} u dx \right|^2 \leq \int_{\Omega} u^2 dx,$$



we obtain

$$\frac{d}{dt} \int_{\Omega} u dx \leq \mu_1 \left( \int_{\Omega} u dx - \frac{1}{|\Omega|} \left| \int_{\Omega} u dx \right|^2 \right). \quad (2.23)$$

Maximum Principle gives us the first inequality. In the same way, we proof the inequality for  $v$ .  $\square$

**Lemma 2.5.** *Let us consider  $p \geq 1$  and  $f_i$  (for  $i = 1, 2$ ) as in (1.13). Under hypotheses (1.8)–(1.12), the following estimates hold:*

$$\frac{1}{p-1} \frac{d}{dt} \int_{\Omega} u^p f_1^{1-p} dx \leq -(\epsilon_u k_{01} + \mu_1) \int_{\Omega} u^{p+1} f_1^{-p} dx + (k_1 + \mu_1) \int_{\Omega} u^p f_1^{1-p} dx$$

and

$$\frac{1}{p-1} \frac{d}{dt} \int_{\Omega} v^p f_2^{1-p} dx \leq -(\epsilon_v k_{02} + \mu_2) \int_{\Omega} v^{p+1} f_2^{-p} dx + (k_2 + \mu_2) \int_{\Omega} v^p f_2^{1-p} dx,$$

where  $k_i$ ,  $k_{0i}$  (for  $i = 1, 2$ ),  $\epsilon_u$  and  $\epsilon_v$  are given in (1.8), (1.9) and (1.5), respectively.

**Proof.** Recall that

$$\frac{d}{dw} f_i = \chi_i(w) f_i, \quad \text{for } i = 1, 2,$$

then, for  $p \geq 1$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p f_1^{1-p} dx &= p \int_{\Omega} u^{p-1} u_t f_1^{1-p} dx + \int_{\Omega} u^p [f_1^{1-p}]' h(u, v, w) dx \\ &= p \int_{\Omega} u^{p-1} u_t f_1^{1-p} dx + (1-p) \int_{\Omega} u^p f_1^{1-p} \chi_1(w) h(u, v, w) dx \\ &= p \int_{\Omega} u^{p-1} (\Delta u - \nabla \cdot (u \chi_1(w) \nabla w)) f_1^{1-p} dx + p \mu_1 \int_{\Omega} u^p f_1^{1-p} (1-u) dx \\ &\quad + (1-p) \int_{\Omega} u^p \chi_1(w) f_1^{1-p} h(u, v, w) dx. \end{aligned} \quad (2.24)$$

Since

$$\begin{aligned} &p \int_{\Omega} u^{p-1} f_1^{1-p} (\Delta u - \nabla \cdot (u \chi_1(w) \nabla w)) dx \\ &= -p \int_{\Omega} \nabla (u^{p-1} f_1^{1-p}) (\nabla u - u \chi_1(w) \nabla w) dx \end{aligned}$$

$$\begin{aligned} &= -p(p-1) \int_{\Omega} u^{p-2} f_1^{1-p} (\nabla u - u \chi_1(w) \nabla w)^2 dx \\ &= -p(p-1) \int_{\Omega} u^{p-2} f_1^{1-p} (e^{\int_0^w \chi_1(s) ds} \nabla (u e^{-\int_0^w \chi_1(s) ds}))^2 dx \end{aligned}$$

we have

$$p \int_{\Omega} u^{p-1} f_1^{1-p} (\Delta u - \nabla \cdot (u \chi_1(w) \nabla w)) dx \leq 0. \tag{2.25}$$

Notice that we can rewrite

$$h(u, v, w) = h(u, v, w) - h(0, v, w) + h(0, v, w) - h(0, 0, w) + h(0, 0, w),$$

by Mean Value Theorem and assumption (1.4) we get

$$h(u, v, w) = \frac{\partial h}{\partial u} \Big|_{(\xi_1, v, w)} u + \frac{\partial h}{\partial v} \Big|_{(0, \xi_2, w)} v + h(0, 0, w)$$

where  $\xi_1 \in (0, u)$  and  $\xi_2 \in (0, v)$ . In view of (1.5), (1.6) and (1.8), we have that

$$-h(u, v, w) \leq -\epsilon_u u - \epsilon_v v - h(0, 0, w) \leq -\epsilon_u u + \frac{k_1}{\chi_1(w)}$$

for  $k_1$  as in (1.8). Then,

$$\begin{aligned} - \int_{\Omega} u^p \chi_1(w) f_1^{1-p} h(u, v, w) dx &\leq -\epsilon_u \int_{\Omega} u^{p+1} \chi_1(w) f_1^{1-p} dx + k_1 \int_{\Omega} u^p f_1^{1-p} dx \\ &\leq -\epsilon_u k_{01} \int_{\Omega} u^{p+1} f_1^{-p} dx + k_1 \int_{\Omega} u^p f_1^{1-p} dx. \end{aligned} \tag{2.26}$$

We also consider the term

$$\int_{\Omega} u^p f_1^{1-p} (1-u) dx \leq \int_{\Omega} u^p f_1^{1-p} dx - \int_{\Omega} u^{p+1} f_1^{-p} dx. \tag{2.27}$$

Thanks to (2.25), (2.26) and (2.27), (2.24) becomes

$$\frac{1}{p-1} \frac{d}{dt} \int_{\Omega} u^p f_1^{1-p} dx \leq -(\epsilon_u k_{01} + \mu_1) \int_{\Omega} u^{p+1} f_1^{-p} dx + (k_1 + \mu_1) \int_{\Omega} u^p f_1^{1-p} dx.$$

In the same fashion we obtain

$$\frac{1}{p-1} \frac{d}{dt} \int_{\Omega} v^p f_2^{1-p} dx \leq -(\epsilon_v k_{02} + \mu_2) \int_{\Omega} v^{p+1} f_2^{-p} dx + (k_2 + \mu_2) \int_{\Omega} v^p f_2^{1-p} dx$$

and the proof is done.  $\square$

**Lemma 2.6.** *Under the same assumptions as in Lemma 2.5, the solutions  $u$ ,  $v$  and  $w$  of (1.1) are uniformly bounded in time by*

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq f_1(\bar{w}) \max\{(k_1 + \mu_1)(\epsilon_u k_{01} + \mu_1)^{-1}, \|u_0\|_{L^\infty(\Omega)}\}, \\ \|v\|_{L^\infty(\Omega)} &\leq f_2(\bar{w}) \max\{(k_2 + \mu_2)(\epsilon_v k_{02} + \mu_2)^{-1}, \|v_0\|_{L^\infty(\Omega)}\}. \end{aligned}$$

**Proof.** We denote by

$$X_p = \int_{\Omega} u^p f_1^{1-p} dx.$$

Notice that, for any  $s > 0$

$$\begin{aligned} \int_{\Omega} u^p f_1^{1-p} dx &\leq \int_{f_1 \leq s(\epsilon_u k_{01} + \mu_1)} u^p f_1^{1-p} dx + \int_{f_1 \geq s(\epsilon_u k_{01} + \mu_1)} u^p f_1^{1-p} dx \\ &\leq s(\epsilon_u k_{01} + \mu_1) \int_{f_1 \leq s(\epsilon_u k_{01} + \mu_1)} u^{p+1} f_1^{-p} dx \\ &\quad + s^{1-p} (\epsilon_u k_{01} + \mu_1)^{1-p} \int_{f_1 \geq s(\epsilon_u k_{01} + \mu_1)} u dx, \end{aligned}$$

i.e.,

$$\int_{\Omega} u^p f_1^{1-p} dx \leq s(\epsilon_u k_{01} + \mu_1) \int_{\Omega} u^{p+1} f_1^{-p} dx + s^{1-p} (\epsilon_u k_{01} + \mu_1)^{1-p} \int_{\Omega} u dx,$$

which implies

$$-(\epsilon_u k_{01} + \mu_1) \int_{\Omega} u^{p+1} f_1^{-p} dx \leq -\frac{1}{s} \int_{\Omega} u^p f_1^{1-p} dx + s^{-p} (\epsilon_u k_{01} + \mu_1)^{1-p} \int_{\Omega} u dx. \quad (2.28)$$

Thanks to (2.28) and Lemma 2.5, we have that

$$\frac{1}{p-1} \frac{d}{dt} X_p \leq \left(k_1 + \mu_1 - \frac{1}{s}\right) X_p + s^{-p} (\epsilon_u k_{01} + \mu_1)^{1-p} \int_{\Omega} u dx. \quad (2.29)$$

By Maximum Principle, taking  $s^{-1} > (k_1 + \mu_1)$ , we obtain

$$\left(\frac{1}{s} - k_1 - \mu_1\right) X_p \leq \max \left\{ s^{-p} (\epsilon_u k_{01} + \mu_1)^{1-p} \int_{\Omega} u dx, X_p^{\frac{1}{p}}(0) \right\},$$

and

$$\lim_{p \rightarrow \infty} X_p^{\frac{1}{p}} \leq \max \{ s^{-1} (\epsilon_u k_{01} + \mu_1)^{-1}, X_p^{\frac{1}{p}}(0) \}.$$

In view of Lemma 2.3 and taking limits as  $s^{-1} \rightarrow (k_1 + \mu_1)$ , it results

$$\|u\|_{L^\infty(\Omega)} \leq f_1(\bar{w}) \max \{ (k_1 + \mu_1) (\epsilon_u k_{01} + \mu_1)^{-1}, \|u_0\|_{L^\infty(\Omega)} \}.$$

In the same way we get

$$\|v\|_{L^\infty(\Omega)} \leq f_2(\bar{w}) \max \{ (k_2 + \mu_2) (\epsilon_v k_{02} + \mu_2)^{-1}, \|v_0\|_{L^\infty(\Omega)} \}$$

which ends the proof.  $\square$

### 3. Asymptotic behavior

The main result of the paper is the following asymptotic behavior for the stationary solutions of system (1.1)–(1.3).

**Theorem 3.1.** *Under assumptions of Theorem 2.1, and either (1.15) or (1.16), (1.17), the unique global solution  $(u, v, w)$  of system (1.1) has the following asymptotic behavior:*

$$\int_{\Omega} |u - 1|^2 dx \rightarrow 0, \quad \int_{\Omega} |v - 1|^2 dx \rightarrow 0, \quad \int_{\Omega} |w - \tilde{w}|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.30)$$

where  $\tilde{w}$  is given by (1.7).

The proof of the theorem is based on an energy estimate, for readers convenience we have split the proof into several lemmata.

**Lemma 3.2.** *Let  $(u, v, w)$  be the solution to (1.1)–(1.3), then, under assumptions of Theorem 3.1, the following inequality holds*

$$\begin{aligned} & \int_{\Omega} |\nabla w|^2 dx + \iint_{\Omega_\infty} (|\nabla u|^2 + |\nabla v|^2) dx dt + \iint_{\Omega_\infty} |\nabla w|^2 dx dt \\ & + \iint_{\Omega_\infty} [\mu_1 u(u - 1)^2 + \mu_2 v(v - 1)^2] dx dt \leq C. \end{aligned}$$

**Proof.** Multiplying by  $u - 1$  and by  $v - 1$  the first two equations in (1.1) and integrating over  $\Omega_T := \Omega \times (0, T)$ , we have

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (u-1)^2 \Big|_0^T dx + \iint_{\Omega_T} |\nabla u|^2 dx dt \\ &= \iint_{\Omega_T} u \chi_1(w) \nabla u \cdot \nabla w dx dt - \mu_1 \iint_{\Omega_T} u(u-1)^2 dx dt, \end{aligned} \quad (3.31)$$

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (v-1)^2 \Big|_0^T dx + \iint_{\Omega_T} |\nabla v|^2 dx dt \\ &= \iint_{\Omega_T} v \chi_2(w) \nabla v \cdot \nabla w dx dt - \mu_2 \iint_{\Omega_T} v(v-1)^2 dx dt. \end{aligned} \quad (3.32)$$

For the variable  $w$ , by (1.1), we compute

$$\nabla w_t = h_u \nabla u + h_v \nabla v + h_w \nabla w.$$

Multiplying the last equation by  $\lambda \nabla w$ , where  $\lambda = \lambda(u, v, w) > 0$  takes arbitrary positive values, we integrate over  $\Omega_T$  to obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \lambda |\nabla w|^2 dx \Big|_0^T &= \iint_{\Omega_T} \lambda h_w |\nabla w|^2 dx dt \\ &+ \iint_{\Omega_T} \lambda h_u \nabla u \cdot \nabla w dx dt + \iint_{\Omega_T} \lambda h_v \nabla v \cdot \nabla w dx dt. \end{aligned} \quad (3.33)$$

Adding (3.31), (3.32) to (3.33) we find that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \lambda |\nabla w(x, T)|^2 dx + \iint_{\Omega_T} (|\nabla u|^2 + |\nabla v|^2) dx dt + \iint_{\Omega_T} (-\lambda h_w) |\nabla w|^2 dx dt \\ & \leq \iint_{\Omega_T} (u \chi_1(w) + \lambda h_u) \nabla u \cdot \nabla w dx dt + \iint_{\Omega_T} (v \chi_2(w) + \lambda h_v) \nabla v \cdot \nabla w dx dt \\ & \quad + O(1), \end{aligned} \quad (3.34)$$

where  $|O(1)| \leq C$ ,  $C$  independent of  $T$ . As a consequence of Lemma 2.6, we know that

$$\frac{1}{2} \left| \int_{\Omega} (u-1)^2 dx \Big|_0^T \right| \leq k, \quad \frac{1}{2} \left| \int_{\Omega} (v-1)^2 dx \Big|_0^T \right| \leq k$$

and by Lemma 2.3, we have that  $u(u-1)^2 \geq 0$ . Applying Schwarz's inequality to the right hand side integral terms in Eq. (3.34) we get the following bounds

$$\begin{aligned} & \iint_{\Omega_T} (u\chi_1(w) + \lambda h_u) \nabla u \cdot \nabla w \, dx \, dt \\ & \leq (1 - \delta) \iint_{\Omega_T} |\nabla u|^2 \, dx \, dt + \frac{1}{4(1 - \delta)} \iint_{\Omega_T} (u\chi_1(w) + \lambda h_u)^2 |\nabla w|^2 \, dx \, dt \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} & \iint_{\Omega_T} (v\chi_2(w) + \lambda h_v) \nabla v \cdot \nabla w \, dx \, dt \\ & \leq (1 - \delta) \iint_{\Omega_T} |\nabla v|^2 \, dx \, dt + \frac{1}{4(1 - \delta)} \iint_{\Omega_T} (v\chi_2(w) + \lambda h_v)^2 |\nabla w|^2 \, dx \, dt \end{aligned} \tag{3.36}$$

for  $0 < \delta < 1$ , small enough. If we impose that there exists  $\lambda > 0$  verifying

$$(u\chi_1(w) + \lambda h_u)^2 + (v\chi_2(w) + \lambda h_v)^2 < 4(-\lambda h_w) \tag{3.37}$$

uniformly for  $(u, v, w)$  then, substituting (3.35)–(3.36) into (3.34) and thanks to (3.37) we deduce

$$\begin{aligned} & \int_{\Omega} |\nabla w|^2 \, dx + \iint_{\Omega_{\infty}} (|\nabla u|^2 + |\nabla v|^2) \, dx \, dt \\ & + \iint_{\Omega_{\infty}} |\nabla w|^2 \, dx \, dt + \iint_{\Omega_{\infty}} \mu_1 u(u - 1)^2 \, dx \, dt + \iint_{\Omega_{\infty}} \mu_2 v(v - 1)^2 \, dx \, dt \leq C. \end{aligned} \tag{3.38}$$

Let us consider the quadratic equation in  $\lambda$

$$(\lambda h_u + u\chi_1(w))^2 + (\lambda h_v + v\chi_2(w))^2 + 4\lambda h_w = 0.$$

The discriminant of the above equation is positive, i.e.,

$$\Delta = [(2uh_u\chi_1(w) + 2vh_v\chi_2(w) + 4h_w)^2 - 4(h_u^2 + h_v^2)(u^2\chi_1^2(w) + v^2\chi_2^2(w))] > 0$$

if (1.15) is satisfied and, in this case, the both roots

$$\begin{aligned} \lambda_{1,2}(u, v, w) = & \frac{1}{2(h_u^2 + h_v^2)} \{ (-2uh_u\chi_1(w) - 2vh_v\chi_2(w) - 4h_w) \\ & \pm [(2uh_u\chi_1(w) + 2vh_v\chi_2(w) + 4h_w)^2 \\ & - 4(h_u^2 + h_v^2)(u^2\chi_1^2(w) + v^2\chi_2^2(w))]^{\frac{1}{2}} \} \end{aligned} \tag{3.39}$$

are positive. Hence (3.37) is verified by choosing any  $\lambda \in (\lambda_2, \lambda_1)$  which ends the proof of the lemma.  $\square$

**Lemma 3.3.** For every  $(u, v, w)$  verifying the system (1.1)–(1.3), we have that

$$\frac{1}{|\Omega|} \int_{\Omega} u dx \rightarrow 1, \quad \frac{1}{|\Omega|} \int_{\Omega} v dx \rightarrow 1.$$

**Proof.** We introduce the notation

$$u^* = \frac{1}{|\Omega|} \int_{\Omega} u dx$$

and integrate the first equation in (1.1) over  $\Omega$  to obtain

$$\frac{d}{dt} u^* = \mu_1 \left( u^* - |u^*|^2 - \frac{1}{|\Omega|} \int_{\Omega} |u - u^*|^2 dx \right). \quad (3.40)$$

We define  $b(t) := (1/|\Omega|) \int_{\Omega} |u - u^*|^2$  which, by Poincaré's inequality, satisfies

$$\int_0^{\infty} b(t) dt = \frac{1}{|\Omega|} \int_{\Omega_{\infty}} |u - u^*|^2 dx dt \leq C \int_{\Omega_{\infty}} |\nabla u|^2 dx dt = C_0 < \infty.$$

We multiply (3.40) by  $(u^* - 1)$  to obtain

$$\frac{d}{dt} (u^* - 1)^2 = \mu_1 \left( u^* - |u^*|^2 - \frac{1}{|\Omega|} \int_{\Omega} |u - u^*|^2 dx \right) (u^* - 1).$$

Thanks to Lemma 2.4, we have that

$$\frac{d}{dt} (u^* - 1)^2 + \mu_1 u^* (u^* - 1)^2 = -\mu_1 b(t) (u^* - 1) \leq c_0 b(t)$$

and after integration

$$\int_0^{\infty} u^* (u^* - 1)^2 dt \leq c < \infty. \quad (3.41)$$

We denote by  $k(t)$

$$k(t) := u^* (u^* - 1)^2 \geq 0.$$

Notice that, by Lemma 2.4,

$$|k'| = |u_t^* (3(u^*)^2 - 2u^* + 1)| = \left| \int_{\Omega} u(1-u) dx \right| |3(u^*)^2 - 2u^* + 1| \leq c_1 < \infty.$$

By Lemma 5.1 (i) of [10] and thanks to (3.41), we conclude that

$$\lim_{t \rightarrow \infty} u^*(u^* - 1)^2 = 0. \tag{3.42}$$

Hence, either  $\lim_{t \rightarrow \infty} u^* = 1$  or  $\lim_{t \rightarrow \infty} u^* = 0$ . It suffices to show that the latter is impossible by obtaining a bound for  $u^*$  from below. Thanks to (3.40) we have that

$$\frac{d}{dt}u^* \leq \mu_1(u^* - |u^*|^2)$$

and after integration

$$u^* \leq \frac{u^*(0)}{u^*(0) + e^{-\mu_1 t}(1 - u^*(0))} = \frac{u^*(0)e^{\mu_1 t}}{u^*(0)e^{\mu_1 t} + 1 - u^*(0)}. \tag{3.43}$$

We consider  $t \leq T$ , where  $T$  is defined as the first  $t > 0$  such that  $u^*(T) = \exp(-\mu_1 \int_0^\infty b(t)dt) \times \min\{1, u^*(0)\}/2$ , if  $\min u^* \leq e^{-\mu_1 \int_0^\infty b(t)dt} \min\{1, u^*(0)\}/2$  and  $T = \infty$  otherwise.

For any  $t \leq T$ , we get

$$\begin{aligned} \frac{d}{dt}u^* &= \mu_1 \left( u^* - (u^*)^2 - \frac{1}{|\Omega|} \int_{\Omega} |u - u^*|^2 dx \right) \\ &\geq \mu_1 \left( u^* - (u^*)^2 - C \int_{\Omega} |\nabla u|^2 dx \right) \\ &\geq \mu_1 \left( u^* - (u^*)^2 - \tilde{C} u^* \int_{\Omega} |\nabla u|^2 dx \right) \\ &= \mu_1 u^* \left( 1 - u^* - \tilde{C} \int_{\Omega} |\nabla u|^2 dx \right) \end{aligned}$$

for

$$\tilde{C} = \frac{C}{\min_{t \in [0, T]} \{u^*\}}.$$

After integration,

$$u^* \geq u^*(0) e^{\mu_1(t - \int_0^t u^*(\tau) d\tau - \int_0^t b(\tau) d\tau)}$$

since

$$\int_0^t b(\tau) d\tau = \int_0^t \int_{\Omega} |\nabla u|^2 dx d\tau \leq \int_0^\infty \int_{\Omega} |\nabla u|^2 dx d\tau \leq C_0$$



and by (3.43) we know that

$$\int_0^t u^*(\tau) d\tau \leq \int_0^t \frac{u^*(0)e^{\mu_1\tau}}{u^*(0)e^{\mu_1\tau} + 1 - u^*(0)} d\tau = \frac{1}{\mu_1} \ln(u^*(0)e^{\mu_1 t} + 1 - u^*(0)),$$

we infer

$$\begin{aligned} u^* &\geq e^{-\mu_1 \int_0^t b(\tau) d\tau} u^*(0) (u^*(0)e^{\mu_1 t} + 1 - u^*(0))^{-1} e^{\mu_1 t} \\ &\geq e^{-\mu_1 \int_0^\infty b(t) dt} \min\{1, u^*(0)\} \end{aligned} \quad (3.44)$$

for  $t < T$ . Inequality (3.44) proves that  $T = \infty$ . Thanks to (3.42) and (3.44), we obtain

$$\lim_{t \rightarrow \infty} |u^* - 1| = 0.$$

In the same way

$$\lim_{t \rightarrow \infty} |v^* - 1| = 0$$

and the proof ends.  $\square$

**Lemma 3.4.** *The following holds*

$$\int_{\Omega} |u - u^*|^2 dx + \int_{\Omega} |v - v^*|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

**Proof.** First we prove that

$$\int_{\Omega} |u - u^*|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.45)$$

and consider the function  $k(t) \geq 0$

$$k(t) := \int_{\Omega} (u(x, t) - u^*(t))^2 dx.$$

By Poincaré's inequality and Lemma 3.2, we find the following upper bound

$$\int_0^\infty k(t) dt \leq \int_0^\infty \int_{\Omega} |\nabla u(x, t)|^2 dx dt \leq C < \infty.$$

Substituting 0,  $T$  by  $t$ ,  $t + s$  in (3.31), we get

$$\left| \int_{\Omega} [(u(x, t+s) - 1)^2 - (u(x, t) - 1)^2] dx \right| \leq C \int_t^{t+s} \int_{\Omega} (|\nabla u(x, \tau)|^2 + |\nabla w(x, \tau)|^2) dx d\tau + \mu_1 \int_t^{t+s} \int_{\Omega} u(u - 1)^2 dx d\tau.$$

Using the relation

$$\begin{aligned} & \int_{\Omega} [(u(x, t+s) - u^*(t+s))^2 - (u(x, t) - u^*(t))^2] dx \\ &= \int_{\Omega} [(u(x, t+s) - 1)^2 - (u(x, t) - 1)^2] dx - 2[u^*(t+s) - u^*(t)] \end{aligned}$$

we have that

$$\int_{\Omega} [(u(x, t+s) - u^*(t+s))^2 - (u(x, t) - u^*(t))^2] dx = \epsilon(t)$$

for

$$\begin{aligned} \epsilon(t) := & C \int_t^{t+s} \int_{\Omega} (|\nabla u(x, \tau)|^2 + |\nabla w(x, \tau)|^2) dx d\tau + \mu_1 \int_t^{t+s} \int_{\Omega} u(u - 1)^2 dx d\tau \\ & + 2|u^*(t+s) - u^*(t)|. \end{aligned}$$

Thanks to [Lemma 3.2](#) and [Lemma 3.3](#), we obtain

$$\epsilon(t) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

By [Lemma 5.1 \(ii\)](#) in [\[10\]](#) (see also [\[15\]](#)), we conclude

$$\int_{\Omega} |u - u^*|^2 dx \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

In the same way we have that

$$\int_{\Omega} |v - v^*|^2 dx \rightarrow 0, \quad \text{as } t \rightarrow \infty$$

which ends the proof.  $\square$

**Proof of Theorem 3.1.** Thanks to [Lemma 3.3](#) and [Lemma 3.4](#) we obtain the first two limits in [\(3.30\)](#). In order to obtain the behavior of  $w$ , we denote by

$$w^*(t) = \frac{1}{|\Omega|} \int_{\Omega} w(x, t) dx \quad (3.46)$$

and introduce the function

$$k(t) = \int_{\Omega} (w(x, t) - w^*(t))^2 dt.$$

With the purpose of obtaining  $k(t) \rightarrow 0$ , as  $t \rightarrow \infty$ , by Lemma 5.1 (i) of [10], it is enough to demonstrate that

$$|k'(t)| \leq C \quad \text{and} \quad \int_0^{\infty} k(t) dt < \infty.$$

The first bound in the previous relation is a consequence of the boundedness of  $w_t$  and the second one is deduced by Poincaré's inequality and (3.38). Thereby, we infer that

$$\int_{\Omega} |w(x, t) - w^*(t)|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.47)$$

To end the demonstration we just need to check that  $w^*(t) \rightarrow \tilde{w}$  as  $t$  goes to infinity. After integration over  $\Omega$  of the third equation in (1.1), we have

$$w_t^* = \frac{1}{|\Omega|} \int_{\Omega} h(u, v, w) dx = h(u^*, v^*, w^*) + \epsilon(t),$$

where

$$|\epsilon(t)|^2 \leq C \int_{\Omega} [|u - u^*|^2 + |v - v^*|^2 + |w - w^*|^2] dx.$$

In view of (3.47) and Lemma 3.4, the last inequality implies

$$\epsilon(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad \text{and} \quad \int_0^{\infty} \epsilon(t)^2 dt < \infty.$$

So,  $w^* - \tilde{w}$  satisfies

$$\frac{d}{dt}(w^* - \tilde{w}) = h(u^*, v^*, w^*) - h(1, 1, w^*) + \epsilon(t).$$

Applying the Mean Value Theorem in the right hand term, we get

$$\frac{d}{dt}(w^* - \tilde{w}) = h_w(1, 1, \xi)(w^* - \tilde{w}) + \epsilon(t)$$

and since  $\inf\{h_w\} < 0$  we have that

$$|w^* - \tilde{w}|^2 \leq c\epsilon^2(t). \tag{3.48}$$

Compounding (3.47) and (3.48), we obtain

$$\int_{\Omega} |w - \tilde{w}|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow \infty, \tag{3.49}$$

and finally, (3.49), Lemma 3.3 and Lemma 3.4 give us the desired result.  $\square$

**Remark 3.5.** Assumptions (1.15) are only used in the proof of Theorem 3.1 to obtain positive values of  $\lambda$  satisfying (3.37). For this purpose, if we demonstrate that

$$(u\chi_1 + \lambda h_u)^2 < -2\lambda h_w \quad \text{and} \quad (v\chi_2 + \lambda h_v)^2 < -2\lambda h_w, \tag{3.50}$$

hence (3.37) is satisfied, so that (3.38) holds. Here is the point where, in Section 1, we introduce the alternative hypotheses (1.17) and (1.16). Let us denote by

$$\lambda_{1,2}(u, v, w) = \frac{-(uh_u\chi_1 + h_w) \pm \sqrt{h_w(2uh_u\chi_1 + h_w)}}{h_u^2}$$

the two roots of the equation in  $\lambda$ ,

$$(u\chi_1 + \lambda h_u)^2 + 2\lambda h_w = 0.$$

Recalling (1.16), we then easily see that the discriminant of the quadratic equation is positive and that both roots are positive. For the second inequality in (3.50) the proof is similar and we find two positive roots, say  $0 < \lambda_3 < \lambda_4$ ,

$$\lambda_{3,4}(u, v, w) = \frac{-(vh_v\chi_2 + h_w) \pm \sqrt{h_w(2vh_v\chi_2 + h_w)}}{h_v^2}.$$

Hence the two inequalities in (3.50) are simultaneously verified by choosing any  $\lambda \in (\lambda_1, \lambda_2) \cap (\lambda_3, \lambda_4)$ . Notice that such  $\lambda(u, v, w)$  exists if  $\lambda_1 \leq \lambda_4$  and  $\lambda_3 \leq \lambda_2$ , which is assured by hypothesis (1.17), where  $\lambda_i = \lambda_i(u, v, w)$ .

**Remark 3.6.** Since any stationary solution  $(\tilde{u}, \tilde{v}, \tilde{w})$  of (1.1), with  $\underline{w} \leq \tilde{w} \leq \bar{w}$ ,

$$0 \leq \tilde{u} \leq (k_1 + \mu_1)(\epsilon_u k_{01} + \mu_1)^{-1}, \quad 0 \leq \tilde{v} \leq (k_2 + \mu_2)(\epsilon_v k_{02} + \mu_2)^{-1},$$

satisfies the estimate

$$\iint_{\Omega_\infty} |\nabla \tilde{u}|^2 dx dt + \iint_{\Omega_\infty} |\nabla \tilde{v}|^2 dx dt + \iint_{\Omega_\infty} |\nabla \tilde{w}|^2 dx dt \leq C < \infty,$$

it follows that such solutions are necessarily constant.

#### 4. Applications

**Example 1.** We consider the case where  $h$  is a linear function,  $h(u, v, w) = u + v - w$  and the chemotactic sensitivities of the species  $\chi_i$ , for  $i = 1, 2$ , are defined by  $\chi_i = \gamma_i / (1 + \gamma_i w)$  for positive constants  $\gamma_i$  verifying

$$\gamma_i < \frac{1}{4} \quad \text{for } i = 1, 2. \quad (4.51)$$

The logistic growth parameters  $\mu_i$  satisfy

$$\mu_i > \frac{3\gamma_i}{1 - 4\gamma_i} \quad (4.52)$$

and the initial data  $(u_0, v_0)$

$$\|u_0\|_{L^\infty(\Omega)} \leq (1 + \mu_1)(\gamma_1 + \mu_1)^{-1}, \quad \|v_0\|_{L^\infty(\Omega)} \leq (1 + \mu_2)(\gamma_2 + \mu_2)^{-1}.$$

1. We have  $h_u = h_v = 1$  and  $h_w = -1$ , so assumptions (1.5) and (1.6) are satisfied for  $\epsilon_u = \epsilon_v = 1$  and a lower bound  $\underline{w} := 0$ .
2. Relation (1.8) is equivalent to

$$w \leq k_i \frac{(1 + \gamma_i w)}{\gamma_i} = k_i w + \frac{k_i}{\gamma_i}. \quad (4.53)$$

For  $k_i = 1$ , (4.53) holds.

3. Taking positive constants  $k_{0i} = \gamma_i$ , for  $i = 1, 2$  hypothesis (1.9) is fulfilled.
4. Notice that  $h(0, 0, 0) = 0$  and

$$h(\bar{u}, \bar{v}, \bar{w}) = \frac{(1 + \mu_1)}{\gamma_1 + \mu_1} (1 + \gamma_1 \bar{w}) + \frac{(1 + \mu_2)}{\gamma_2 + \mu_2} (1 + \gamma_2 \bar{w}) - \bar{w}.$$

For any upper bound  $\bar{w}$

$$\bar{w} \geq \max_{i=1,2} \left\{ \frac{2(1 + \mu_i)}{\mu_i(1 - 2\gamma_i) - \gamma_i} \right\}, \quad (4.54)$$

with  $\mu_i$  and  $\gamma_i$  as in (4.51) and (4.52), we have  $h(\bar{u}, \bar{v}, \bar{w}) \leq 0$ .

5. Remains to be studied what restrictions are necessary to be fulfilled (1.15). By Remark 1.1, a sufficient condition to have asymptotic stability is

$$2\gamma_i(1 + \gamma_i \bar{w}) \frac{1 + \mu_i}{\gamma_i + \mu_i} < 1 \quad \text{for } i = 1, 2,$$

which is equivalent to take  $\bar{w}$  such that

$$\bar{w} < \frac{\mu_i(1 - 2\gamma_i) - \gamma_i}{2\gamma_i^2(1 + \mu_i)}. \quad (4.55)$$

Thanks to (4.52) there exists  $\bar{w}$  satisfying (4.54) and (4.55). Under the above restrictions, we can apply Theorem 2.1 and Theorem 3.1, thus we have global existence of the solution  $(u, v, w)$  of (1.1) and the constant solution is the only steady state solution satisfying

$$0 \leq u \leq \bar{u}, \quad 0 \leq v \leq \bar{v}, \quad \underline{w} \leq w \leq \bar{w} \tag{4.56}$$

and is furthermore asymptotically stable. The unique stationary solution of system (1.1) satisfying (4.56) is given by  $(u, v, w) := (1, 1, 2)$  and

$$\lim_{t \rightarrow \infty} \int_{\Omega} |u - 1|^2 dx = \lim_{t \rightarrow \infty} \int_{\Omega} |v - 1|^2 dx = \lim_{t \rightarrow \infty} \int_{\Omega} |w - 2|^2 dx = 0.$$

**Example 2.** Take  $\chi_i$ , with  $i = 1, 2$  to be two positive constants and  $h(u, v, w) = \alpha u + v - \mu w$ , such that  $\mu$  verifies

$$\mu > \max_{i=1,2} \frac{2e^B \mu_i \chi_i}{B \mu_i + B \chi_i (\alpha - 2e^B)} > 0 \tag{4.57}$$

with positive  $\alpha, \mu_i, i = 1, 2$ , where  $B$  is

$$B := \frac{2(\sqrt{2} - 1)}{\max\{1, \alpha\}}. \tag{4.58}$$

We consider the cases where the logistic parameters  $\mu_i$  satisfy

$$\mu_i > (2e^B - \alpha) \chi_i \quad \text{for } i = 1, 2. \tag{4.59}$$

In order to obtain the global existence of the solutions of (1.1) and to prove that the constant solution is the only steady state, and is furthermore asymptotically stable, we have to verify that assumptions (1.5), (1.6), (1.8)–(1.10) and (1.15) are fulfilled.

Thus, we choose  $\underline{w} = 0$  and  $\bar{w}$  in the interval

$$\bar{w} \in \left[ \max_{i=1,2} \left\{ \frac{2e^B \mu_i}{\mu \mu_i + \mu \chi_i (\alpha - 2e^B)} \right\}, \min_{i=1,2} B \chi_i^{-1} \right] \tag{4.60}$$

for  $B$  defined in (4.58).

1. We have  $h_u = \alpha, h_v = 1$  and  $h_w = -\mu$ , so in (1.5) and (1.6) we take  $\epsilon_u = \alpha$  and  $\epsilon_v = 1$ .
2. Assumption (1.8) is verified for

$$k_i = \mu \bar{w} \chi_i.$$

3. Inequality (1.9) is satisfied for

$$k_{0i} = \chi_i, \quad \text{for } i = 1, 2.$$

4. Notice that  $h(0, 0, 0) = 0$  and

$$h(\bar{u}, \bar{v}, \bar{w}) = e^{\chi_1 \bar{w}} \frac{\mu \bar{w} \chi_1 + \mu_1}{\alpha \chi_1 + \mu_1} + e^{\chi_2 \bar{w}} \frac{\mu \bar{w} \chi_2 + \mu_2}{\chi_2 + \mu_2} - \mu \bar{w}.$$

As in the previous example, after some computations, we find that if  $w$  is greater than or equal to the left interval extreme of (4.60), then  $h(\bar{u}, \bar{v}, \bar{w}) \leq 0$ .

5. In order to obtain the stability, we need

$$\bar{w} < B \chi_i^{-1}.$$

Therefore, for every  $\bar{w}$  as in (4.60), for all initial data  $(u_0, v_0, w_0)$  of (1.1) satisfying

$$0 \leq u_0 \leq \frac{\mu \bar{w} \chi_1 + \mu_1}{\alpha \chi_1 + \mu_1}, \quad 0 \leq v_0 \leq \frac{\mu \bar{w} \chi_2 + \mu_2}{\chi_2 + \mu_2} \quad \text{and} \quad 0 \leq w_0 \leq \bar{w}$$

such that

$$u_0 \neq 0, \quad \text{and} \quad v_0 \neq 0$$

the above conditions (4.57)–(4.60) are sufficient to guarantee the global existence and to apply Theorem 3.1. The unique stationary solution of system (1.1) satisfying

$$0 \leq u \leq \bar{u}, \quad 0 \leq v \leq \bar{v}, \quad \underline{w} \leq w \leq \bar{w},$$

is given by

$$(u, v, w) := \left( 1, 1, \frac{\alpha + 1}{\mu} \right)$$

and it satisfies

$$\lim_{t \rightarrow \infty} \int_{\Omega} |u - 1|^2 dx = \lim_{t \rightarrow \infty} \int_{\Omega} |v - 1|^2 dx = \lim_{t \rightarrow \infty} \int_{\Omega} \left| w - \frac{1 - \alpha}{\mu} \right|^2 dx = 0.$$

**Example 3.** We consider  $h$  given by

$$h(u, v, w) = \frac{u + v + \alpha}{1 + \gamma w} - \beta, \quad \text{and} \quad \chi_i = \frac{\gamma_i}{1 + \gamma_i w} \quad \text{for } i = 1, 2$$

with positive constants  $\alpha, \beta, \gamma, \gamma_1$  and  $\gamma_2$  satisfying

$$\alpha \geq \beta \tag{4.61}$$

$$\gamma \leq \min_{i=1,2} \{\gamma_i, \gamma_2\}. \tag{4.62}$$

We take  $\underline{w} = 0$  and  $\bar{w}$  in the interval

$$\bar{w} \in [\bar{w}_1, \bar{w}_2] \cap \left[ \frac{\alpha}{\beta\gamma_i} + \frac{\alpha - \beta}{\beta\gamma}, +\infty \right) \tag{4.63}$$

where  $\bar{w}_1$  and  $\bar{w}_2$  are the positive roots of the second order equation in  $w$

$$2(1 + \gamma_i w)^2 \left( \gamma_i \frac{\beta^2}{4\alpha} + \mu_i \right) \gamma_i < \alpha\gamma(\gamma_i + \mu_i(\gamma w + 1)). \tag{4.64}$$

In order to obtain such results we have to verify that assumptions (1.5), (1.6), (1.8)–(1.10) and (1.16) hold.

1. We have

$$h_u = h_v = \frac{1}{1 + \gamma w}, \quad h_w = -\frac{(u + v + \alpha)\gamma}{(1 + \gamma w)^2},$$

so (1.5) and (1.6) fulfill for

$$\epsilon_u = \epsilon_v = \frac{1}{1 + \gamma \bar{w}}.$$

2. Taking

$$k_i = \gamma_i \frac{\beta^2}{4\alpha} \quad \text{and} \quad k_{0i} = \gamma_i, \quad \text{for } i = 1, 2,$$

we have (1.8) and (1.9).

3. Relation (1.16) is equivalent to

$$2(1 + \gamma_i \bar{w}) \frac{\gamma_i \frac{\beta^2}{4\alpha} + \mu_i}{\frac{\gamma_i}{\gamma \bar{w} + 1} + \mu_i} < \frac{\alpha\gamma}{\gamma_i} \quad \text{for } i = 1, 2 \tag{4.65}$$

and taking  $w \in [\bar{w}_1, \bar{w}_2]$ , we get the desired result.

4. Notice that  $h(0, 0, 0) = \alpha - \beta$  and thanks to assumption (4.61), we have that  $h(0, 0, 0) \geq 0$ . Relation (1.10) is equivalent to

$$h(\bar{u}, \bar{v}, \bar{w}) = 2(1 + \gamma_i \bar{w}) \frac{\gamma_i \frac{\beta^2}{4\alpha} + \mu_i}{\frac{\gamma_i}{\gamma \bar{w} + 1} + \mu_i} + \alpha - \beta(\gamma \bar{w} + 1) \leq 0, \quad \text{for } i = 1, 2.$$

Once we have (4.65), then, for any  $\bar{w} \geq \alpha/(\beta\gamma_i) + (\alpha - \beta)/(\beta\gamma)$ , we obtain  $h(\bar{u}, \bar{v}, \bar{w}) \leq 0$  and (1.10) is satisfied.

Now that all the required hypotheses are verified, as we have discussed in Remark 1.1, we can apply Theorem 2.1 and Theorem 3.1 and the solution  $(u, v, w)$  of (1.1) is globally uniformly bounded and satisfies



$$\lim_{t \rightarrow \infty} \int_{\Omega} |u - 1|^2 dx = \lim_{t \rightarrow \infty} \int_{\Omega} |v - 1|^2 dx = \lim_{t \rightarrow \infty} \int_{\Omega} |w - \tilde{w}|^2 dx = 0$$

for  $\tilde{w}$  such that  $h(1, 1, \tilde{w}) = 0$ , i.e.,

$$\tilde{w} = \frac{2 + \alpha - \beta}{\beta\gamma},$$

for initial data satisfying

$$0 \leq u_0 \leq \frac{\frac{\gamma_1 \beta^2}{4\alpha} + \mu_1}{\frac{1}{1+\gamma\bar{w}}\gamma_1 + \mu_1}, \quad 0 \leq v_0 \leq \frac{\frac{\gamma_2 \beta^2}{4\alpha} + \mu_2}{\frac{1}{1+\gamma\bar{w}}\gamma_2 + \mu_2} \quad \text{and} \quad 0 \leq w_0 \leq \bar{w}$$

and

$$u_0 \neq 0, \quad \text{and} \quad v_0 \neq 0.$$

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