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ON A COMPARISON METHOD TO REACTION-DIFFUSION SYSTEMS AND ITS APPLICATIONS TO CHEMOTAXIS

Mihaela Negreanu

Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain

J. Ignacio Tello

Departamento de Matemática Aplicada, ETSI SI
Universidad Politécnica de Madrid
28031 Madrid, Spain

Abstract. In this paper we consider a general system of reaction-diffusion equations and introduce a comparison method to obtain qualitative properties of its solutions. The comparison method is applied to study the stability of homogeneous steady states and the asymptotic behavior of the solutions of different systems with a chemotactic term. The theoretical results obtained are slightly modified to be applied to the problems where the systems are coupled in the differentiated terms and / or contain nonlocal terms. We obtain results concerning the global stability of the steady states by comparison with solutions of Ordinary Differential Equations.

1. **Introduction.** In this paper we apply a comparison method to parabolic systems in order to study the stability of homogeneous steady states and the asymptotic behavior of the solutions of different problems. Comparison methods based on upper and lower solutions have been applied to a large number of reaction-diffusion systems as the extensive literature shows (see for instance [10] and reference therein). Comparison method based on ODE′s systems have already used in the last decades, see for instance [7], [15] and [8] for chemotactic systems of two PDE′s and extended to parabolic-parabolic-elliptic chemotactic systems in [16], [9] and [14]. We study a reaction-diffusion parabolic system *"weakly coupled"* (i.e. the system is coupled only in the terms which are not differentiated) in a bounded domain $\Omega \subset I\!R^n$ with regular boundary $\partial\Omega$. We denote by Ω_T the set defined by $(x,t) \in \Omega \times (0,T)$ and consider the problem

$$\begin{aligned} &u_t - \mathcal{L}(u) = g(u), && (x,t) \in \Omega_T, \\ &\mathcal{B}u = 0, && (x,t) \in \partial\Omega \times (0,T), \\ &u = u_0, && x \in \Omega, \end{aligned} \tag{1}$$

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with $u := (u_1, \ldots, u_m)$, $\mathcal{L} := (\mathcal{L}_1, \ldots, \mathcal{L}_m)$. $\mathcal{L}_i$ is a second-order linear elliptic differential operator of the form

$$\mathcal{L}_i(u_i) := \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a^i_{jk}(x,t) \frac{\partial}{\partial x_k} u_i - \sum_{j=1}^{n} b^i_j(x,t) \frac{\partial}{\partial x_j} u_i - c^i(x,t) u_i,$$

with measurable coefficients a^i_{jk}, $b^i := (b^i_1, \ldots, b^i_n)$ and c^i, satisfying the ellipticity condition

$$\sum_{j,k=1}^{n} a^i_{jk}(x,t)\xi_j\xi_k \geq \Lambda|\xi|^2 \qquad \text{for some } \Lambda > 0 \text{ and } \xi \in I\!R^n, \tag{2}$$

and

$$a^i_{jk} \in C(\overline{\Omega}_T); \qquad |a^i_{jk}|,\ |b^i_j|,\ |c^i| < \Lambda_1 \qquad \text{for some } \Lambda_1 > 0. \tag{3}$$

The second equation in (1) gives the boundary conditions where the operator $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_m)$ is defined by

$$\mathcal{B}_i(u) := (A^i \nabla u_i) \cdot \nu \quad \text{where} \quad A^i := (a^i_{jk}(x,t))_{j,k=1\ldots n}$$

and ν is the exterior unit normal on $\partial\Omega$. We assume that

$$g \text{ is Loc. Lipschitz continuous in } u \text{ and Hölder continuous in } x \text{ and } t. \tag{4}$$

and the initial data

$$u_0 \in (W^{2,p}(\Omega))^m \quad \text{for} \quad p > n \tag{5}$$

satisfies the boundary condition.

If there exist functions $\overline{u} := (\overline{u}_1, \ldots, \overline{u}_m)$ and $\underline{u} := (\underline{u}_1, \ldots, \underline{u}_m)$ such that

$$\overline{u}, \underline{u} \in [L^p(0,T : W^{2,p}(\Omega)) \cap W^{1,p}(0,T : L^p(\Omega))]^m$$

for some $p > n$, satisfying the following system of $2m-$ parabolic inequalities

$$\begin{aligned}
&\frac{\partial}{\partial t}\overline{u}_i - \mathcal{L}_i(\overline{u}_i) \geq \underset{\substack{\underline{u}_j \leq u_j \leq \overline{u}_j \text{ for } j \neq i \\ u_i = \overline{u}_i}}{\text{maximum}} g_i(u), \quad (x,t) \in \Omega_T, \\
&\mathcal{B}_i \overline{u}_i \geq 0 \qquad (x,t) \in \partial\Omega \times (0,T), \\
&\frac{\partial}{\partial t}\underline{u}_i - \mathcal{L}_i(\underline{u}_i) \leq \underset{\substack{\underline{u}_j \leq u_j \leq \overline{u}_j \text{ for } j \neq i \\ u_i = \underline{u}_i}}{\text{minimum}} g_i(u), \quad (x,t) \in \Omega_T, \\
&\mathcal{B}_i \underline{u}_i \leq 0, \qquad (x,t) \in \partial\Omega \times (0,T), \\
&\underline{u_i}(x,0) \leq u_i(x,0) \leq \overline{u_i}(x,0), \quad x \in \Omega,
\end{aligned} \tag{6}$$

then, there exists at least one solution to the problem, satisfying

$$\underline{u}_i \leq u_i \leq \overline{u}_i, \qquad \text{for } i = 1, \ldots, m.$$

A similar comparison method for regular coefficients can be found in [10]. For cooperative systems, i.e., under assumptions

$$\frac{\partial g_i}{\partial u_j} \geq 0 \quad \text{for} \quad i \neq j,$$

the solutions have also the property of order preserving, i.e., if

$$u_i(0,x) \leq v_i(0,x) \quad \text{for} \quad x \in \Omega,$$

then

$$u_i(t,x) \leq v_i(t,x)$$

for the maximal interval of existence, see for instance [11]. The order preserving property is not satisfied for general nonlinear systems of parabolic equations.

In this work we apply a comparison method to two different systems of partial differential equations arising from biological processes. The results in the literature can not be applied directly to the problems since the equations are coupled in the differentiated terms or contain nonlocal expressions.

- The first problem we study is the asymptotic stability of homogeneous steady state of a two competitive populations of biological species, both of which are attracted chemotactically by the same signaling substance. The chemical is introduced by a forcing term f and it is decoupled from the two previous equations.

$$\begin{cases} u_t + A_1(u) = -\chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1-u-a_1 v), & x \in \Omega,\ t>0, \\ v_t + A_2(v) = -\chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1-a_2 u - v), & x \in \Omega,\ t>0, \\ -\Delta w + \lambda w = f(x,t), \quad x \in \Omega,\ t>0, \end{cases} \tag{7}$$

where μ_1 and μ_2 represent the growth rates of species u and v; the terms $-\mu_1 u^2$ and $-\mu_1 v^2$ represent the inhibition effects that u and v have on the growth of u and v, respectively; the term $-\mu_1 a_1 uv$ measures the influence of v on the growth of u; and $-\mu_2 a_2 vu$ the inhibiting effect of u on the growth of v. In (7) A_i (for $i=1,2$) are second order differential operator of Leray-Lions type (see [6]) defined by

$$A_1 u := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a^1_{jk}(x,u,\nabla u)\frac{\partial u}{\partial x_k},\ A_2 v := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a^2_{jk}(x,v,\nabla v)\frac{\partial u}{\partial x_k}$$

satisfying

H1. $A_1 : V_1 \to V_1'$ and $A_2 : V_2 \to V_2'$ are continuous for $V_1 := W^{1,p}(\Omega)$ and $V_2 := W^{1,q}(\Omega)$.

H2. There exists $C>0$ such that $\|A_1 u\|_{V_1'} \leq C\|u\|_{V_1}$ and $\|A_2 v\|_{V_2'} \leq C\|v\|_{V_2}$ for $u \in V_1$, $v \in V_2$.

H3. A_i (for $i=1,2$) are strongly monotone in the following interpolation sense: there exist $\theta_i \in (0,1]$ and $c>0$ such that for all $\phi_1, \phi_2 \in V_i$ we have

$$c\|\phi_1 - \phi_2\|_{V_i} \leq < A_i\phi_1 - A_i\phi_2, \phi_1 - \phi_2 >^{\theta_i} (\|\phi_1\|_{V_i} + \|\phi_2\|_{V_i})^{1-\theta_i}.$$

H4. $< A_i\phi_1 - A_i\phi_2, \phi_1-\phi_2 > \geq 0$ for all $\phi_1, \phi_2 \in V_i$ $(i=1,2)$.

Notice that $A_1 = \Delta_p$ and $A_2 = \Delta_q$ satisfy (H1)-(H4).

We consider the following boundary conditions in a bounded and regular domain $\Omega \subset I\!R^n$, for $n \geq 1$

$$\sum_{j,k=1}^{n} a^1_{jk} \frac{\partial u}{\partial x_k}\nu_j = \sum_{j,k=1}^{n} a^2_{jk}\frac{\partial v}{\partial x_k}\nu_j = \sum_{j=1}^{n}\frac{\partial w}{\partial j}\nu_j = 0, \qquad x \in \partial\Omega,\ t>0 \tag{8}$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega \tag{9}$$

satisfying (8). We also assume that $f \in C^{\alpha,1+\beta}_{x,t}(\overline{\Omega} \times [0,T])$, f is uniformly bounded and satisfies

$$\|f - \frac{1}{|\Omega|}\int_\Omega f\|_{L^\infty(\Omega)} \to 0, \qquad \text{as} \quad t \to \infty. \tag{10}$$

The problem (7) is considered as a first step to study the control problem, i.e. to find f in a suitable space such that the solutions have a desired behavior. The system with constant diffusion coefficients where w satisfies the linear elliptic equation

$$-\Delta w + \lambda w = k_1 u + k_2 v, \qquad x \in \Omega, \ t > 0,$$

have been already analyzed in [16] for a range of parameters, see also [9].

- The second application considers a degenerate reaction-diffusion system with nonlocal sources modelling a cooperative system of two biological species. We consider the unknown densities "u" and "v" satisfying the system

$$\begin{cases} u_t = \Delta u^{r_1} - \chi_1 \nabla(u\nabla w) - au + \int_\Omega |v|^p, & x \in \Omega, \ t > 0, \\ v_t = \Delta v^{r_2} - \chi_2 \nabla(v\nabla w) - bv + \int_\Omega |u|^q, & x \in \Omega, \ t > 0, \\ -\Delta w + \lambda w = u + v, \qquad x \in \Omega \end{cases}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \qquad x \in \partial\Omega, \quad t > 0,$$

for $r_1, r_2 \geq 1$, $p, q, a, b > 0$ and $|\Omega| = 1$. We obtain results on the asymptotic behavior and blow-up for some range of initial data and parameters. The system for $\chi_1 = \chi_2 = 0$ has been also used to model the temperature of two substances in a combustible mixture (see for instance [12], [13] or [17] and reference therein).

We assume that the initial data satisfy

$$u_0, v_0 \in C^{2+\alpha}_x(\Omega) \tag{11}$$

and

$$u_0, v_0 > 0 \quad \text{in} \quad \overline{\Omega}, \qquad \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \quad \text{in} \quad \partial\Omega. \tag{12}$$

In order to study the mentioned problems above, we first analyze the weakly coupled general system. For the completeness of the result, we detail the comparison method for the problem (1) in Section 2. These results can not be applied directly to the problems studied in Section 3 due to the nonlocal terms or nonlinear diffusion coefficient that we treat separately.

2. **Comparison method.**

In this section we study the comparison method for the problem (1) employing Schauder's fixed point theorem. We assume that there exists $T > 0$ such that the solutions of problem (6) exist in $(0, T)$.

The main purpose of this section is the following theorem:

Theorem 2.1. *Under hypothesis (2)–(5), if the initial data satisfy*

$$\underline{u}_i(x,0) \leq u_i(x,0) \leq \overline{u}_i(x,0), \tag{13}$$

the unique solution of (1) fulfills

$$\underline{u}_i \leq u_i \leq \overline{u}_i, \qquad \text{for } i = 1 \ldots m, \tag{14}$$

where $\underline{u}_i$ and $\overline{u}_i$ satisfy (6).

Proof. Let A be defined as follows:

$$A := \left\{ u \in [C^0(\Omega_T)]^m \ \text{ such that } \underline{u}_i \leq u_i \leq \overline{u}_i, \quad \text{ for } \ i = 1 \dots m \right\}.$$

Notice that A is a bounded set in $[C^0(\Omega_T)]^m$. Let $J : A \to A$ be defined by

$$J(\tilde{u}) = u$$

where u is the solution to the problem

$$\left\{ \begin{array}{ll} u_t - \mathcal{L}(u) = \tilde{g}(u), & (x,t) \in \Omega_T \\ \\ \mathcal{B}u = 0, & (x,t) \in \partial\Omega \times (0,T) \end{array} \right. \tag{15}$$

with $\tilde{g}(u) := (\tilde{g}_1(u), \dots, \tilde{g}_m(u))$ for

$$\tilde{g}_i(u) = g_i(\tilde{u}_1, \dots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \dots, \tilde{u}_m).$$

For simplicity, in order to prove that J has a fixed point, we divide the proof into several steps.

Step 1. We see first that J is well defined for T small enough.

We linearize the problem and apply a fixed point argument to prove the existence and uniqueness of solution of (15). Due to parabolic regularity and assumptions (2), (3) and (5) the solution to the linear decoupled problem belongs to

$$[L^p(0,T : W^{2,p}(\Omega)) \cap W^{1,p}(0,T : L^p(\Omega))]^m$$

for some $p > n$ (see Remark 48.3 in [11]). Thanks to (4) we obtain using a fixed point method that

$$u \in [L^p(0,T : W^{2,p}(\Omega)) \cap W^{1,p}(0,T : L^p(\Omega))]^m,$$

which implies

$$u \in [C^0(\Omega_T)]^m, \qquad \text{for } T \text{ small enough.}$$

Step 2. $u \in A$.

We denote by

$$\overline{U}_i(x,t) := u_i(x,t) - \overline{u}_i(x,t), \quad \underline{U}_i(x,t) := u_i(x,t) - \underline{u}_i(x,t),$$

$$\overline{g}_i := \underset{\substack{\underline{u}_j \leq u_j \leq \overline{u}_j \text{ for } j \neq i \\ u_i = \overline{u}_i}}{\text{maximum}} g_i(u), \ \underline{g}_i := \underset{\substack{\underline{u}_j \leq u_j \leq \overline{u}_j \text{ for } j \neq i \\ u_i = \underline{u}_i}}{\text{minimum}} g_i(u), \quad \text{for } t > 0,$$

and the standard positive and negative part functions:

$$(s)_+ = \left\{ \begin{array}{ll} s & \text{if } s \geq 0, \\ 0 & \text{otherwise} \end{array} \right. \qquad (s)_- = (-s)_+.$$

With these notations, resting (6) and (1), we obtain the following PDE system

$$\frac{\partial}{\partial t}\overline{U}_i - \mathcal{L}_i \overline{U}_i \leq \tilde{g}_i(u) - \overline{g}_i, \tag{16}$$

$$\frac{\partial}{\partial t}\underline{U}_i - \mathcal{L}_i \underline{U}_i \geq \tilde{g}_i(u) - \underline{g}_i, \tag{17}$$

$$\mathcal{B}_i \overline{U}_i \leq 0, \tag{18}$$

$$\mathcal{B}_i \underline{U}_i \geq 0, \tag{19}$$

$$\underline{U}_i(x,0) = u_i(x,0) - \underline{u}_i(x,0), \qquad \overline{U}_i(x,0) = u_i(x,0) - \overline{u}_i(x,0). \tag{20}$$

We take $(\overline{U}_i)_+$ as test function in (16), i.e., we multiply by $(\overline{U}_i)_+$ and integrate by parts over Ω to obtain, after some computations:

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int_\Omega(\overline{U}_i)_+^2+\int_\Omega\Lambda|\nabla(\overline{U}_i)_+|^2+\int_\Omega b^i\nabla(\overline{U}_i)_+(\overline{U}_i)_++\int_\Omega c^i(\overline{U}_i)_+^2\le\\ \int_\Omega\left(\tilde{g}_i(u)-\overline{g}_i\right)(\overline{U}_i)_+.\end{gathered}\tag{21}$$

Thanks to (3), we estimate the third term in the left part of inequality (21) as follows

$$\int_\Omega b^i(x,t)\nabla(\overline{U}_i)_+(\overline{U}_i)_+\le k_1\int_\Omega(\overline{U}_i)_+^2+\frac{\Lambda}{4}\int_\Omega|\nabla(\overline{U}_i)_+|^2.$$

In order to simplify the term $\tilde{g}_i(u)-\overline{g}_i$, we add $\pm g_i(\tilde{u}_1,\dots,\tilde{u}_{i-1},\overline{u}_i,\tilde{u}_{i+1},\dots,\tilde{u}_m)$, i.e., $\tilde{g}_i(u)-\overline{g}_i=$

$$\tilde{g}_i(u)-g_i(\tilde{u}_1,\dots,\tilde{u}_{i-1},\overline{u}_i,\tilde{u}_{i+1},\dots,\tilde{u}_m)+g_i(\tilde{u}_1,\dots,\tilde{u}_{i-1},\overline{u}_i,\tilde{u}_{i+1},\dots,\tilde{u}_m)-\overline{g}_i.$$

Thanks to the definition of $\overline{g}_i$ it follows

$$g_i(\tilde{u}_1,\dots,\tilde{u}_{i-1},\overline{u}_i,\tilde{u}_{i+1},\dots,\tilde{u}_m)-\overline{g}_i\le 0,$$

and by Mean Value Theorem

$$\tilde{g}_i(u)-\overline{g}_i\le\left.\frac{\partial}{\partial u_i}\tilde{g}_i(u)\right|_{u_i=\xi}\overline{U}_i$$

for some $\xi\in(\overline{u}_i,u_i)\cup(u_i,\overline{u}_i)$. By assumption (4) we have

$$\int_\Omega\left(\tilde{g}_i(u)-\overline{g}_i\right)(\overline{U}_i)_+\le k_i\int_\Omega(\overline{U})_+^2.$$

Therefore, from (21) it follows

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\overline{U}_i)_+^2+\frac{\Lambda}{4}\int_\Omega|\nabla(\overline{U}_i)_+|^2\le k_i\int_\Omega(\overline{U}_i)_+^2,\tag{22}$$

for $t\in(0,T)$.

In the same fashion as before we multiply equation (17) by $(\underline{U}_i)_-$ to derive, using similar computations as those in (21), the inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\underline{U}_i)_-^2+\frac{\Lambda}{4}\int_\Omega|\nabla(\overline{U}_i)_-|^2\le k_i'\int_\Omega(\underline{U}_i)_-^2.\tag{23}$$

Finally, adding (22) and (23) we see that

$$\frac{d}{dt}\int_\Omega\sum_{i=1}^m\left((\overline{U}_i)_+^2+(\underline{U}_i)_-^2\right)\le K\int_\Omega\sum_{i=1}^m\left((\overline{U}_i)_+^2+(\underline{U}_i)_-^2\right),\tag{24}$$

for all $t\in(0,T)$ and $K=\max_{i=1,\dots,m}\{k_i,k_i'\}$. By (13), the initial data satisfy $(\overline{U}_i)_+=(\underline{U}_i)_-=0$ at $t=0$ and we apply Gronwall's Lemma to achieve

$$(\overline{U}_i)_+=(\underline{U}_i)_-=0,$$

which proves $u\in A$ for T small enough. The solution is extended to $(0,T_{max})$, where $(0,T_{max})$ is the maximum interval of definition of $\overline{u}_i$ and $\underline{u}_i$.

Step 3. Compactness.

Since $u_i\in L^\infty(0,T:L^\infty(\Omega))$ we have that $u_t-\mathcal{L}u\in L^q(0,T:L^q(\Omega))$ for any $q<\infty$, by (5) and [11] (Remark 48.3 (ii)) we have that

$$u_i\in Y_p:=L^p(0,T:W^{2,p}(\Omega))\cap H^{1,p}(0,T:L^p(\Omega)),$$

for some $p > n$. Since $Y_p \subset C^0((0,T)\times\Omega)$ is a compact embedding for $T < \infty$ and A is a bounded set, we have, by Schauder fixed point theorem that J has at least one fixed point in $\overline{A}$, the solution to the problem. Uniqueness is a consequence of the regularity of g. □

3. Applications.

3.1. Problem 1.

First we study a system of type (7), (8) and (9) consisting on three partial differential equations modelling the spatio-temporal behavior of two competitive populations of biological species (u, v), both of which are attracted chemotactically by the same signal substance "w".

In [16], the authors have considered the case $f = u + v$ and they obtained that, when $0 \le a_1 < 1$ and $0 \le a_2 < 1$, the system possesses a uniquely determined spatially homogeneous positive equilibrium (u^*, v^*), globally asymptotically stable within a certain nonempty range of the logistic growth coefficients μ_1 and μ_2. In our case the parameters λ, χ_1, χ_2, μ_1 and μ_2 are assumed to be positive.

We rewrite system (7) as follows

$$\begin{cases} u_t + A_1 u = -\chi_1 \nabla w \cdot \nabla u + \chi_1 u(f(x,t) - \lambda w) + \mu_1 u(1 - u - a_1 v), & \text{in } \Omega_T, \\ v_t + A_2 v = -\chi_2 \nabla w \cdot \nabla v + \chi_2 v(f(x,t) - \lambda w) + \mu_2 v(1 - a_2 u - v), & \text{in } \Omega_T, \\ -\Delta w + \lambda w = f(x,t), \quad \text{in } \Omega_T, \end{cases}$$

where A_i satisfy (H1)-(H4).

Remark 1. We consider the following auxiliary problem

$$u_t + A_1 u = f_1, \qquad v_t + A_2 v = f_2 \qquad \text{in} \quad \Omega_T$$

with the boundary conditions given by (8) for any $f_1 \in L^p(0,T : W^{-1,p}(\Omega))$ and $f_2 \in L^q(0,T : W^{-1,q}(\Omega))$. Since A_i satisfy assumptions (H1)-(H4) we obtain the existence and uniqueness of solutions $(u,v) \in (L^p(0,T : W^{1,p}(\Omega)), L^q(0,T : W^{1,q}(\Omega)))$. The proof is similar to the result in Derlet and Takáč [6] Proposition 2.1, where homogeneous Neumann boundary conditions are considered. An standard fixed point argument in f_i gives the existence of weak solutions. Uniqueness of solutions is a consequence of assumption (H4) and regularity of the reaction terms.

Since Theorem 2.1 can not be applied directly to the system due to the differentiated terms, we consider the following lemma.

Lemma 3.1. *Under hypothesis (2)–(5) and (H1)-(H4) we have that if there exist $(\overline{u}, \underline{u})$, $(\overline{v}, \underline{v})$ such that*

$$\begin{cases} \overline{u}_t + A_1 \overline{u} \le -\chi_1 \nabla w \cdot \nabla \overline{u} + \chi_1 \overline{u}(f(x,t) - \lambda w) + \mu_1 \overline{u}(1 - \overline{u} - a_1 \underline{v}), & \textit{in } \Omega_T, \\ \underline{u}_t + A_1 \underline{u} \ge -\chi_1 \nabla w \cdot \nabla \underline{u} + \chi_1 \underline{u}(f(x,t) - \lambda w) + \mu_1 \underline{u}(1 - \underline{u} - a_1 \overline{v}), & \textit{in } \Omega_T, \\ \overline{v}_t + A_2 \overline{v} \le -\chi_2 \nabla w \cdot \nabla \overline{v} + \chi_2 \overline{v}(f(x,t) - \lambda w) + \mu_2 \overline{v}(1 - a_2 \underline{u} - \overline{v}), & \textit{in } \Omega_T, \\ \underline{v}_t + A_2 \overline{v} \ge -\chi_2 \nabla w \cdot \nabla \underline{v} + \chi_2 \underline{v}(f(x,t) - \lambda w) + \mu_2 \underline{v}(1 - a_2 \overline{u} - \underline{v}), & \textit{in } \Omega_T, \\ -\Delta w + \lambda w = f(x,t), \quad \textit{in } \Omega_T, \end{cases}$$

satisfying the boundary conditions (8) and the inequalities

$$\underline{u}(0) \le u(x,0) \le \overline{u}(0), \qquad \underline{v}(0) \le v(x,0) \le \overline{v}(0) \tag{25}$$

then, the unique solution of (7) fulfills

$$\underline{u} \le u \le \overline{u}, \quad \underline{v} \le v \le \overline{v}. \tag{26}$$

Proof. We denote by

$$\overline{U}(x,t) := u(x,t) - \overline{u}(t), \quad \underline{U}(x,t) := u(x,t) - \underline{u}(t),$$

and

$$\overline{V}(x,t) := v(x,t) - \overline{v}(t), \quad \underline{V}(x,t) := v(x,t) - \underline{v}(t).$$

The proof follows the steps of Theorem 2.1 except for the step 1, which should be replaced by Remark 1. The rest of the proof is similar to Theorem 2.1 except for the differentiated terms which we treat in the following way

$$< A_1 u - A_1 \overline{u}, (\overline{U})_+ >= \int_\Omega \sum_{j,k=1}^n a^1_{jk}(x,u,\nabla u) \frac{\partial}{\partial x_j} \overline{U} \frac{\partial}{\partial x_k} (\overline{U})_+ \geq 0,$$

$$\begin{aligned} \textstyle\int_\Omega (\overline{U})_+ \nabla \overline{U} \cdot \nabla w &= \tfrac{1}{2} \textstyle\int_\Omega \nabla (\overline{U})^2_+ \cdot \nabla w \\ &= -\tfrac{1}{2} \textstyle\int_\Omega (\overline{U})^2_+ \Delta w \\ &= \tfrac{1}{2} \textstyle\int_\Omega (\overline{U})^2_+ (f - \lambda w) \\ &\leq c \textstyle\int_\Omega (\overline{U})^2_+ . \end{aligned}$$

The rest of the terms

$$< A_1 u - A_1 \underline{u}, (\underline{U})_- >, \; < A_2 v - A_1 \overline{v}, (\overline{V})_+ >, \; < A_2 v - A_2 \underline{v}, (\underline{U})_- >,$$

$$\int_\Omega (\underline{U})_- \nabla \underline{U} \cdot \nabla w, \quad \int_\Omega (\overline{V})_+ \nabla \overline{V} \cdot \nabla w \quad \text{and} \quad \int_\Omega (\underline{V})_- \nabla \underline{V} \cdot \nabla w$$

are treated in the same fashion. The rest of the proof reproduces the steps of Theorem 2.1. □

The main result of this section is as follows

Theorem 3.2. *Let $n \geq 1$ and $\Omega \subset I\!R^n$ be a bounded domain with smooth boundary. Assume that λ, χ_i, μ_i and a_i are positive for $i = 1, 2$ and f satisfies (10). For all positive initial data $u_0 \in C^0(\bar\Omega)$ and $v_0 \in C^0(\bar\Omega)$, the solution (u, v) to (7) is bounded and satisfies*

$$\|u(\cdot,t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v^*\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty \tag{27}$$

where the constant steady (u^*, v^*) takes one of the following values

- $(u^*, v^*) = \left(\dfrac{1-a_1}{1-a_1a_2}, \dfrac{1-a_2}{1-a_1a_2} \right)$ *if* $0 \leq a_i < 1, \; i = 1, 2,$
- $(u^*, v^*) = (1, 0)$, *if* $\quad 0 \leq a_1 < 1 < a_2,$
- $(u^*, v^*) = (0, 1)$, *if* $\quad 0 \leq a_2 < 1 < a_1.$

If $1 < a_i$, for $i = 1, 2$ and $f = 0$, we have that the solution satisfies (27) for (u^*, v^*) given by

- $(u^*, v^*) = (1, 0) \quad$ *if* $\quad \max\limits_{x \in \Omega} v_0 < \min\limits_{x \in \Omega} u_0 \dfrac{a_2 - 1}{a_1 - 1},$
- $(u^*, v^*) = (0, 1) \quad$ *if* $\quad \min\limits_{x \in \Omega} v_0 > \max\limits_{x \in \Omega} u_0 \dfrac{a_2 - 1}{a_1 - 1}.$

In order to prove the theorem we introduce a system of Ordinary Differential Equations for the upper and lower solutions

$$\begin{cases} \overline{u}' = \overline{u}\ \Big[\mu_1 - \mu_1\overline{u} - \mu_1 a_1 \underline{v} + \chi_1 \sup\limits_{x\in\Omega}\{f(x,t) - \lambda w\}\Big], \quad t>0,\\ \underline{u}' = \underline{u}\ \Big[\mu_1 - \mu_1\underline{u} - \mu_1 a_1 \overline{v} + \chi_1 \inf\limits_{x\in\Omega}\{f(x,t) - \lambda w\}\Big], \quad t>0,\\ \overline{v}' = \overline{v}\ \Big[\mu_2 - \mu_2 a_2\underline{u} - \mu_2 \overline{v} + \chi_2 \sup\limits_{x\in\Omega}\{f(x,t) - \lambda w\}\Big], \quad t>0,\\ \underline{v}' = \underline{v}\ \Big[\mu_2 - \mu_2 a_2\overline{u} - \mu_2 \underline{v} + \chi_2 \inf\limits_{x\in\Omega}\{f(x,t) - \lambda w\}\Big], \quad t>0, \end{cases}$$

with positive initial data $\overline{u}_0$, $\underline{u}_0$, $\overline{v}_0$ and $\underline{v}_0$. Following the outline of Theorem 2.1, we analyze the stability of the system. Denoting by

$$M(t) := \sup_{x\in\Omega}\{f(x,t) - \lambda w(x,t)\}, \qquad m(t) := \inf_{x\in\Omega}\{f(x,t) - \lambda w(x,t)\}, \tag{28}$$

we observe that the system can be expressed as two independent systems of equations:

$$\begin{cases} \overline{u}' = \overline{u}\ \Big[\mu_1 - \mu_1\overline{u} - \mu_1 a_1\underline{v} + \chi_1 M(t)\Big],\\ \underline{v}' = \underline{v}\ \Big[\mu_2 - \mu_2 a_2\overline{u} - \mu_2\underline{v} + \chi_2 m(t)\Big], \end{cases}$$
$$\begin{cases} \underline{u}' = \underline{u}\ \Big[\mu_1 - \mu_1\underline{u} - \mu_1 a_1\overline{v} + \chi_1 m(t)\Big],\\ \overline{v}' = \overline{v}\ \Big[\mu_2 - \mu_2 a_2\underline{u} - \mu_2\overline{v} + \chi_2 M(t)\Big]. \end{cases} \tag{29}$$

We shall study the solutions' properties of the first system in (29). The other one is symmetric and all the results obtained for the first one are valid for the second system.

We analyze (29) taking into account that $M(t) \to 0$ and $m(t) \to 0$ as $t \to \infty$ and we can justify that it is natural to work in this framework. If we denote by

$$\overline{f}(t) = \sup_{x\in\Omega} f(x,t), \qquad \underline{f}(t) = \inf_{x\in\Omega} f(x,t),$$

by Maximum Principle, in (7), we know that

$$\underline{f}(t) \le \lambda w(x,t) \le \overline{f}(t), \qquad \forall\ x\in\Omega. \tag{30}$$

By assumption (10) and taking into account (30), we obtain

$$\lim_{t\to\infty} \|w - \frac{1}{\lambda|\Omega|}\int_\Omega f\|_{L^\infty(\Omega)} = 0$$

and therefore

$$M(t) := \sup_{x\in\Omega}\{f(x,t) - \lambda w(x,t)\} \to 0,\ m(t) := \inf_{x\in\Omega}\{f(x,t) - \lambda w(x,t)\} \to 0. \tag{31}$$

Notice that, by integration in $-\Delta w + \lambda w = f$, we have that $\displaystyle\int_\Omega (f - \lambda w) = 0$, for all $t > 0$ and it implies that $M(t) \ge 0$ and $m(t) \le 0$.

System (29) has been widely studied in the literature. For the autonomous case (i.e. $M(t) = m(t) = 0$) Braun in [5] details the asymptotic behavior depending on the parameters. To the author's knowledge, the asymptotic properties of the solutions for the general case of the nonautonomous system (29) have been studied for the first time by Ahmad in [1] and [4], under certain assumptions for the coefficients, i.e., always assumed to be bounded, continuous and nonnegative. Using only simple arguments based on differential inequalities and standard theorems concerning continuity of solutions of differential equations with respect to initial conditions and parameters, it is possible to find optimal bounds and convergence results for the solutions of (29).

For readers′ convenience we quote the results in [5], [1] and [4] used in the proof of Theorem 3.2.

- **Case I.** $M = m = 0$.

  In this case we have an autonomous system (commonly called an autonomous Lotka Volterra system) as a model of competition between two species

$$\begin{cases} u_1' = u_1\ [\mu_1 - \mu_1 u_1 - \mu_1 a_1 u_2], \\ u_2' = u_2\ [\mu_2 - \mu_2 a_2 u_1 - \mu_2 u_2]. \end{cases} \tag{32}$$

For $M = m = 0$, both systems in (29) coincide, and for simplicity, we had denoted their solutions by ($u_1 = \overline{u} = \underline{u}$, $u_2 = \underline{v} = \overline{v}$). The equilibrium points of (32) are $(0,0)$, $(1,0)$, $(0,1)$ and

$$(u^*, v^*) = \left(\frac{1-a_1}{1-a_1 a_2}, \frac{1-a_2}{1-a_1 a_2} \right). \tag{33}$$

(**I.a**) In [5] and the references therein, it is proved that a phase plane analysis of this autonomous case shows that the conditions

$$0 \leq a_i < 1 \qquad \text{for} \qquad i = 1,2 \tag{34}$$

are necessary and sufficient for the existence of a unique stable equilibrium point (u^*, v^*) of system (32) given by (33), such that both components are positive and it globally attracts all solutions with initial values in the open first quadrant of the (u_1, u_2) plane.

If (34) fails, then, generally the model either predicts that one of the competitors always becomes extinct while the other persists, or that the outcome of the competition depends on the initial data. Another feature of the system (32) is that if (u_1, u_2) is a solution which is nonnegative in both components then the change of variables $(v_1, v_2) = (u_1, -u_2)$ converts (32) to a cooperative system. Cooperative systems are well known to be order preserving so if (u_1, u_2) and $(\tilde{u}_1, \tilde{u}_2)$ are nonnegative solutions to (32) with $u_1(0) \geq \tilde{u}_1(0)$ and $u_2(0) \geq \tilde{u}_2(0)$ then $u_1(t) \geq \tilde{u}_1(t)$ and $u_2(t) \geq \tilde{u}_2(t)$ for all $t > 0$.

If conditions (34) fail two different cases occur:

(**I.b**) Assume that in (32)

$$0 \leq a_1 < 1 < a_2. \tag{35}$$

Then, any solution $(u_1(t), u_2(t))$ of (32) with $u_1(t_0) > 0$, $u_2(t_0) > 0$ for some $t_0 > 0$, verifies $(u_1(t), u_2(t)) \to (1, 0)$, as $t \to \infty$. This is sometimes referred to as the principle of competitive exclusion. That is, u_2 approaches zero as t approaches infinity for every solution $(u_1(t), u_2(t))$ of (32) with $u_1(t_0) > 0$. By symmetry, if we assume that

$$0 \leq a_2 < 1 < a_1, \tag{36}$$

then $(u_1(t), u_2(t)) \to (0,1)$, as $t \to \infty$.
(**I.c**) Assume that

$$a_1 > 1 \qquad \text{and} \qquad a_2 > 1. \tag{37}$$

Then,
1. The equilibrium solution $(0,0)$ of (32) is unstable.
2. The equilibrium solutions $(0,1)$ and $(1,0)$ of (32) are locally asymptotically stable.
3. The equilibrium solution (u^*, v^*) of (32) defined by (33) is a saddle point.

- **Case II.** $M(t) \to 0$ and $m(t) \to 0$. We consider our problem as a particular case of a Lotka-Volterra system (see a large and complete study of this problem in [1] and [4])

$$\begin{cases} u_1' = u_1\ [\tilde{\mu}_1(t) - \mu_1 u_1 - \mu_1 a_1 u_2], \\ \\ u_2' = u_2\ [\tilde{\mu}_2(t) - \mu_2 a_2 u_1 - \mu_2 u_2]. \end{cases} \tag{38}$$

Given a function $g(t)$, which is bounded above and below by positive constants for $t_0 \le t < \infty$, we let g_L and g_M denote $\inf\limits_{t \ge t_0} g(t)$ and $\sup\limits_{t \ge t_0} g(t)$, respectively. The coefficients in (38) are always assumed to be bounded, continuous, and nonnegative, therefore the solution exists in $(0, \infty)$ and it is uniformly bounded.
(**II.a**) In [1] it was shown that if the coefficients $\tilde{\mu}_i(\cdot)$ are bounded below by positive constants, then, if the inequalities

$$\tilde{\mu}_{1L}\mu_2 > a_1\mu_1\tilde{\mu}_{2M} \quad \text{and} \quad \tilde{\mu}_{2L}\mu_1 > a_2\mu_2\tilde{\mu}_{1M}, \tag{39}$$

hold, then there exists a solution $u^*(t) = (u_1^*(t), u_2^*(t))$ such that the inequalities

$$\begin{aligned} 0 &< \frac{\tilde{\mu}_{1L}\mu_2 - a_1\mu_1\tilde{\mu}_{2M}}{\mu_1\mu_2 - a_1a_2\mu_1\mu_2} \equiv s_1 \le u_1^*(t) \le r_1 \equiv \frac{\tilde{\mu}_{1M}\mu_2 - a_1\mu_1\tilde{\mu}_{2L}}{\mu_1\mu_2 - a_1a_2\mu_1\mu_2}, \\ 0 &< \frac{\mu_1\tilde{\mu}_{2L} - \tilde{\mu}_{1M}a_2\mu_2}{\mu_1\mu_2 - a_1a_2\mu_1\mu_2} \equiv r_2 \le u_2^*(t) \le s_2 \equiv \frac{\mu_1\tilde{\mu}_{2M} - \tilde{\mu}_{1L}a_2\mu_2}{\mu_1\mu_2 - a_1a_2\mu_1\mu_2} \end{aligned} \tag{40}$$

hold for $t_0 \le t < \infty$. These bounds are optimal since, in the autonomous space, the upper bound for each component coincides with the lower bound for that component. Another important result obtained in [1] is: *if conditions (40) hold and $(u_1(t), u_2(t))$ and $(v_1(t), v_2(t))$ are any two solutions of (38) such that $u_k(t_1) > 0$, $v_k(t_1) > 0$ for $k = 1, 2$ and for some $t_1 \ge t_0$, then $u_1(t) - v_1(t) \to 0$ and $u_2(t) - v_2(t) \to 0$ as $t \to \infty$. Thus it follows that if $(u_1(t), u_2(t))$ is any solution of (38) with both components positive at some time and ε is any arbitrary positive number, then*

$$s_1 - \varepsilon < u_1(t) < r_1 + \varepsilon, \qquad r_2 - \varepsilon < u_2(t) < s_2 + \varepsilon$$

for sufficiently large t.

Notice that if $u_1(t)$ and $u_2(t)$ are positive solutions of the logistic equations $u_1'(t) = u_1(t)[\tilde{\mu}_1(t) - \mu_1 u_1(t)]$ and $u_2'(t) = u_2(t)[\tilde{\mu}_2(t) - \mu_2 u_2(t)]$, respectively, then the pairs $(u_1(t), 0)$ and $(0, u_2(t))$ are solutions of (38). Moreover, the open first quadrant in the (u_1, u_2)-plane is invariant in the sense that if $(u_1(t), u_2(t))$ is a solution of (38) with $u_1(0) > 0$ and $u_2(0) > 0$ then $u_1(t) > 0$ and $u_2(t) > 0$ for all $t > 0$.

If conditions (39) fail the following occurs:

(**II.b**) Under the assumption that the inequalities

$$\tilde{\mu}_{1L}\mu_2 > a_1\mu_1\tilde{\mu}_{2M}, \quad \text{and} \quad \mu_1\tilde{\mu}_{2M} \le a_2\mu_2\tilde{\mu}_{1L} \tag{41}$$

hold (these are equivalent to (35) in the constant coefficients case), the main result proved in [2] gives us exact information about the development of the solutions of (38): *If $(u_1(t), u_2(t))$ is any solution of (38) such that $u_1(t_0) > 0$ and $u_2(t_0) > 0$ for some t_0 in $(-\infty, \infty)$, then $u_2(t) \to 0$ and $u_1(t) - \hat{u}(t) \to 0$ as $t \to \infty$, where $\hat{u}(t)$ is the solution of the logistic equation*

$$\hat{u}'(t) = \hat{u}(t)\Big[\tilde{\mu}_1(t) - \mu_1\hat{u}(t)\Big], \tag{42}$$

such that $\delta_1 \le \hat{u}(t) \le \delta_2$ on $(-\infty, \infty)$ where δ_1 and δ_2 are any numbers satisfying the inequalities $0 < \delta_1 < \tilde{\mu}_{1L}/\mu_1 \le \tilde{\mu}_{1M}/\mu_1 < \delta_2$.

Of course, a similar result, where the role of u_1 and u_2 are interchanged, will hold if the inequalities in (41) are replaced by

$$\tilde{\mu}_{1M}\mu_2 \le a_1\mu_1\tilde{\mu}_{2L}, \qquad \mu_1\tilde{\mu}_{2L} > a_2\mu_2\tilde{\mu}_{1M}. \tag{43}$$

An extension of this principle for nonautonomous systems was given in [2], where it was shown that similar algebraic inequalities imply that there can be no coexistence of the two species. One of them will be driven to extinction while the other will stabilize at a certain solution of a logistic equation.

In [3] it was considered a somewhat more general system, and given a further extension of the result by introducing a sufficient condition that is implied by (41) (and (43) respectively). If in (38) the coefficients verify the following conditions: $\tilde{\mu}_i(t) = \mu_i + f_i(t)$ are continuous and $f_i(t) \le c_i\, e^{-\gamma_i t}$ where c_i and γ_i are positive constants and we do not assume the growth rate μ_i is positive, if (35) holds then if $(u_1(t), u_2(t))$ is any solution of (38) such that $u_1(t_0) > 0$, $u_2(t_0) > 0$, then $u_2(t) \to 0$ exponentially and $u_1(t) \to \hat{u}(t)$ as $t \to \infty$, where $\hat{u}(t)$ is the unique positive solution of

$$\hat{u}'(t) = \hat{u}(t)\Big[\mu_1 - \mu_1\hat{u}(t)\Big].$$

Proof of Theorem 3.2. Observe that all the coefficients for the general case (38) are always assumed to be bounded, continuous, and nonnegative: in our case, taking into account the continuity of f and the definitions of M and m (31), we confirm that these hypothesis are verified. For t_0 large enough, taking into account (31), we can choose the positive parameters μ_i, χ_i, $i = 1, 2$, such that

$$\mu_i + \chi_i m(t) \ge 0, \qquad \forall\, t \ge t_0. \tag{44}$$

This will be the framework along the rest of this section and all the results are valid under conditions (44). Taking into account the structure of time depending coefficients, by (31), in limit, we rediscover from conditions corresponding to non-autonomous case the same conditions of the constant coefficients case. Thanks to above results concerning the asymptotic behavior of (29) and (38) we have

- Under assumption (34) for positive parameters λ, χ_1, χ_2, μ_1 and μ_2, any solution $(\overline{u}(t), \underline{v}(t))$, $(\underline{u}(t), \overline{v}(t))$ of (29), with positive initial data satisfies

  $$\overline{u}(t) \to u^*, \qquad \underline{u}(t) \to u^*, \qquad \underline{v}(t) \to v^*, \qquad \overline{v}(t) \to v^* \quad \text{as} \quad t \to \infty,$$

  with (u^*, v^*) given by (33). In other words, under assumptions (34), we have that all solutions $(\overline{u}(t), \underline{v}(t))$ and $(\underline{u}(t), \overline{v}(t))$ of (29), with both pairs

$(\overline{u}(t_0), \underline{v}(t_0))$ and $(\underline{u}(t_0), \overline{v}(t_0))$ positive, ultimately approach the equilibrium solution (33). Therefore, applying (14) and Lemma 3.1 we conclude

$$||u(\cdot,t) - u^*||_{L^\infty(\Omega)} + ||v(\cdot,t) - v^*||_{L^\infty(\Omega)} \to 0 \qquad as \quad t \to \infty. \tag{45}$$

- Assume the parameters λ, χ_1, χ_2, μ_1 and μ_2 are positive and (35) holds. For any solution $(\overline{u}(t), \underline{v}(t))$, $(\underline{u}(t), \overline{v}(t))$ of system (29), with positive initial data, we have

$$\overline{u}(t) \to 1, \qquad \underline{u}(t) \to 1, \qquad \underline{v}(t) \to 0, \qquad \overline{v}(t) \to 0 \quad \text{as} \quad t \to \infty.$$

This is equivalent to saying that the species $\overline{v}$ and $\underline{v}$ become extinct if $\overline{u}(t_0) > 0$ and $\underline{u}(t_0) > 0$ and thanks to Theorem 2.1 and Lemma 3.1 we have

$$||u(\cdot,t) - 1||_{L^\infty(\Omega)} + ||v(\cdot,t)||_{L^\infty(\Omega)} \to 0 \qquad as \quad t \to \infty. \tag{46}$$

By symmetry if $a_1 > 1$ and $0 \le a_2 < 1$ we have

$$||u(\cdot,t)||_{L^\infty(\Omega)} + ||v(\cdot,t) - 1||_{L^\infty(\Omega)} \to 0 \qquad as \quad t \to \infty. \tag{47}$$

- If (37) is satisfied, $M = m = 0$ (i.e. $f \equiv 0$) and the initial data (u_0, v_0) of (7) verifies

$$\underline{v}_0 = \min_x v_0 > \overline{u}_0 \frac{a_2 - 1}{a_1 - 1} = \max_x u_0 \frac{a_2 - 1}{a_1 - 1}, \tag{48}$$

then

$$(\underline{u}, \overline{v}) \to (0,1), \qquad (\overline{u}, \underline{v}) \to (0,1) \qquad \text{as} \quad t \to \infty \tag{49}$$

and therefore, by Theorem 2.1 and Lemma 3.1

$$||u(\cdot,t)||_{L^\infty(\Omega)} + ||v(\cdot,t) - 1||_{L^\infty(\Omega)} \to 0 \qquad \text{as} \quad t \to \infty. \tag{50}$$

The proof of the asymptotic behavior of $(\underline{u}, \overline{u}, \underline{v}, \overline{v})$ is similar to Theorem 6 in [Braun [5], Chapter 4, section 11] see also exercise 6 in [5] p. 456.

On the other hand, if the initial data (u_0, v_0) of (7) satisfies

$$\overline{v}_0 = \max_x v_0 < \underline{u}_0 \frac{a_2 - 1}{a_1 - 1} = \min_x u_0 \frac{a_2 - 1}{a_1 - 1}, \tag{51}$$

then

$$(\underline{u}, \overline{v}) \to (1,0), \qquad (\overline{u}, \underline{v}) \to (1,0) \qquad as \quad t \to \infty \tag{52}$$

and therefore

$$||u(\cdot,t) - 1||_{L^\infty(\Omega)} + ||v(\cdot,t)||_{L^\infty(\Omega)} \to 0 \qquad as \quad t \to \infty. \tag{53}$$

Observe that under restriction (37), the stability of (u^*, v^*) and $(0,0)$ in the case **(I.c)**, the ODE analysis doesn't give us more information about the problem (7).

Since

$$\tilde{\mu}_i(t) \longrightarrow \mu_i \quad \text{for} \quad i = 1, 2$$

the solution of the non-autonomous system converges to one of the equilibrium solutions of the autonomous system defined in Theorem 3.2 as a consequence of (10). The equilibrium solution where the solution converges is determined by the sign of the following limits

$$\lim_{t\to\infty} \left(\tilde{\mu}_{1M}\mu_2 - \mu_1 a_1 \tilde{\mu}_{2L}\right), \qquad \lim_{t\to\infty} \left(\tilde{\mu}_{2L}\mu_1 - a_2\mu_2\tilde{\mu}_{1M}\right)$$

which coincide with the signs of $1 - a_1$ and $1 - a_2$, respectively. $\square$

3.2. **Problem 2.** In this subsection we consider a system that describes the evolution of a cooperative system of two biological species (u, v) which satisfy

$$\left\{\begin{array}{ll} u_t = d_1\Delta u^{r_1} - \chi_1\nabla(u\nabla w) - au + \int_\Omega |v|^p, & x\in\Omega,\ t>0, \\ v_t = d_2\Delta v^{r_2} - \chi_2\nabla(v\nabla w) - bv + \int_\Omega |u|^q, & x\in\Omega,\ t>0, \\ -\Delta w + \lambda w = u + v \qquad x\in\Omega,\ t>0, & \end{array}\right. \tag{54}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \qquad x\in\partial\Omega, \quad t>0, \tag{55}$$

where $\Omega \subset I\!\!R^n$ is a bounded domain with smooth boundary such that $|\Omega| = 1$ and

$$a>0, \quad b>0, \quad r_1\ge 1, \qquad r_2\ge 1, \qquad p\ge 1 \quad \text{and} \quad q\ge 1.$$

The problem, where $\chi_1 = \chi_2 = 0$ has been studied by several authors (see for instance [12], [13] and [17] and reference therein) where the critical exponents and blow-up rate of solutions are given. In [17] is proved a result concerning the local existence of solutions that are extended to a maximal interval of existence $(0, T_{max})$ such that

$$\lim_{t\to T_{max}} \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + T_{max} = \infty.$$

We consider the asymptotic behaviour of the solutions by using the comparison method presented in Section 2 for positive parameters χ_1 and χ_2. We also provide results on the stability of steady states. Since nonlinear diffusion coefficients are not considered in Section 2 we can not directly apply Theorem 2.1 to (54)-(55) for $r_1 > 1$ or $r_2 > 1$. Nevertheless, it is possible to use a similar argument for the particular case of spatially homogeneous sub- and super- solution $\underline{u}$, $\underline{v}$, $\overline{u}$ and $\overline{v}$. Notice that the nonlocal term is not considered neither in Theorem 2.1.

The problem can be expressed as follows

$$\left\{\begin{array}{ll} u_t = d_1\Delta u^{r_1} - \chi_1\nabla u\cdot\nabla w + \chi_1 u(u+v-\lambda w) - au + \int_\Omega |v|^p, & x\in\Omega,\ t>0, \\ v_t = d_2\Delta v^{r_2} - \chi_2\nabla v\cdot\nabla w + \chi_2 u(u+v-\lambda w) - bv + \int_\Omega |u|^q, & x\in\Omega,\ t>0, \\ -\Delta w + \lambda w = u+v \qquad x\in\Omega,\ t>0. & \end{array}\right. \tag{56}$$

In order to study the stability of the system, we first introduce the next lemma.

Lemma 3.3. *Under assumptions (11), (12), there exists $T_{max} > 0$ such that the unique solution to the problem (54)-(55) satisfies*

$$u(x,t), v(x,t) > 0, \qquad in \quad x\in\Omega, \quad t<T$$

where T_{max} satisfies

$$\lim_{t\to T_{max}} \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + t = \infty.$$

Proof. The existence of solutions follows a fixed point argument for the decoupled problem. Let us consider the set A, as the functions $(\psi_1, \psi_2) \in [C^{\alpha,\frac{\alpha}{2}}(\Omega_T)]^2$ satisfying

$$e^{-(|a|+|b|)t}\min\left\{\inf_{x\in\Omega} u_0, \inf_{x\in\Omega} v_0\right\} \le \psi_i \le 2\max\left\{\sup_{x\in\Omega} u_0, \sup_{x\in\Omega} v_0\right\}$$

and let $J : A \to [C^{\alpha,\frac{\alpha}{2}}(\Omega_T)]^2$ be given by $J(\tilde{u},\tilde{v}) = (u,v)$ where (u,v) is the solution to the problem

$$\begin{cases} u_t = d_1 r_1 \nabla \cdot \tilde{u}^{r_1-1}\nabla u - \chi_1 \nabla u \nabla w - au + \int_\Omega |\tilde{v}|^p + \chi_1 u(\tilde{u}+\tilde{v}-\lambda w), & \text{in } \Omega_T, \\ v_t = d_2 r_2 \nabla \cdot \tilde{v}^{r_2-1}\nabla v - \chi_2 \nabla v \nabla w - bv + \int_\Omega |\tilde{u}|^q + \chi_2 v(\tilde{u}+\tilde{v}-\lambda w), & \text{in } \Omega_T, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \end{cases}$$

where w satisfies

$$-\Delta w + \lambda w = \tilde{u} + \tilde{v} \qquad \text{in } \Omega_T,$$

with Neumann boundary conditions. Linear PDEs Theory (see [11] Remark 48.3 (ii) page 439) gives us the existence and uniqueness of solutions in

$$[L^p(0,T:W^{2,p}(\Omega)) \cap W^{1,p}(0,T:L^p(\Omega))]^2 \subset W^{1,p}(\Omega_T)^2, \quad \text{for some } p > n.$$

For T small enough we have that the solution remains in A. Since $[W^{1,p}(\Omega_T)]^2 \hookrightarrow [C^{\alpha,\frac{\alpha}{2}}(\Omega_T)]^2$ is a compact embedding, a Schauder fixed point Theorem allows us to obtain the existence of solutions in Ω_T for T small enough. Uniqueness is a consequence of the monotonicity of the differential operators, Lemma 3.3 and assumption $p,q \geq 1$. We now may extend the solution to a maximal interval $(0, T_{max})$ where the solution remains bounded and positive, since $(e^{-kt}\inf_{x\in\Omega} u_0, e^{-kt}\inf_{x\in\Omega} v_0)$ is a sub-solution to the problem (for some $k := k(u_0, v_0, a, b)$) we have that the solution remains positive and T_{max} satisfies

$$\lim_{t\to T_{max}} \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + T_{max} = \infty.$$

$\square$

Remark 2. Notice that

$$u, v \in L^p(0,T:W^{2,p}(\Omega)) \cap W^{1,p}(0,T:L^p(\Omega))]^2 \subset C^{\alpha,\frac{\alpha}{2}}(\Omega_{T_{max}}).$$

In order to apply a comparison method, we introduce the following system of equations

$$\overline{u}' = -a\overline{u} + \overline{v}^p + \chi_1 \overline{u}(\overline{u} + \overline{v} - \underline{u} - \underline{v}), \quad t > 0, \tag{57}$$

$$\underline{u}' = -a\underline{u} + \underline{v}^p + \chi_1 \underline{u}(\underline{u} + \underline{v} - \overline{u} - \overline{v}) \quad t > 0, \tag{58}$$

$$\overline{v}' = -b\overline{v} + \overline{u}^q + \chi_2 \overline{v}(\overline{u} + \overline{v} - \underline{u} - \underline{v}), \quad t > 0, \tag{59}$$

$$\underline{v}' = -b\underline{v} + \underline{u}^q + \chi_2 \underline{v}(\underline{u} + \underline{v} - \overline{u} - \overline{v}), \quad t > 0. \tag{60}$$

Notice that the solutions of the system are non negative provided the initial data $\overline{u}_0$, $\underline{u}_0$, $\overline{v}_0$ and $\underline{v}_0$ are non negative.

Theorem 3.4. *The unique solution to (54)-(55) satisfies*

$$\underline{u} \leq u \leq \overline{u}, \qquad \underline{v} \leq v \leq \overline{v}, \qquad (x,t) \in \Omega \times (0, T_{max}), \tag{61}$$

where $\overline{u}$, $\underline{u}$, $\overline{v}$ and $\underline{v}$ are the solutions to (57)-(60) and T_{max} is the maximum time of existence.

Proof. The proof is similar to the proof of Theorem 2.1 except for the step 1, which is replaced by Lemma 3.3, and the treatment of the following integrals:

$$-\int_\Omega \overline{U}_+ \Delta u^{r_1} = -r_1 \int_\Omega \overline{U}_+ div(u^{r_1-1}\nabla u) = r_1 \int_\Omega u^{r_1-1}|\nabla \overline{U}_+|^2 \geq 0.$$

In the same way we have

$$\int_\Omega \underline{U}_- \Delta u^{r_1} = r_1 \int_\Omega u^{r_1-1} |\nabla \underline{U}_-|^2 \geq 0,$$

$$-\int_\Omega \overline{V}_+ \Delta v^{r_2} = r_2 \int_\Omega v^{r_2-1} |\nabla \overline{V}_+|^2 \geq 0$$

and

$$\int_\Omega \underline{V}_- \Delta v^{r_2} = r_2 \int_\Omega v^{r_2-1} |\nabla \underline{V}_-|^2 \geq 0.$$

The nonlocal terms are treated as follows: since u and $\overline{u}$ are non-negative, we have that, by Mean Value Theorem, $u^p - \overline{u}^p = p\xi^{p-1}(u - \overline{u})$ for some positive function ξ. Notice that ξ is a bounded function for any $t < T_{max}$. Therefore

$$\overline{U}_+ \int_\Omega (v^p - \overline{v}^p) = \overline{U}_+ \int_\Omega p\xi^{p-1}(v - \overline{v}) \leq \overline{U}_+ \int_\Omega p\xi^{p-1} \overline{V}_+ \leq c(\xi, \Omega) \overline{U}_+ \|\overline{V}_+\|_{L^1(\Omega)}$$

and

$$\begin{aligned} \textstyle\int_\Omega \left(\overline{U}_+ \int_\Omega (v^p - \overline{v}^p)\right) &\leq c(\xi, \Omega) \|\overline{U}_+\|_{L^1(\Omega)} \|\overline{V}_+\|_{L^1(\Omega)} \\ &\leq c_2(\xi_1, \Omega) \left(\|\overline{U}_+\|^2_{L^2(\Omega)} + \|\overline{V}_+\|^2_{L^2(\Omega)} \right). \end{aligned}$$

In the same way we have

$$-\int_\Omega \left(\underline{U}_- \int_\Omega (v^p - \underline{v}^p) \right) \leq \int_\Omega \left(\underline{U}_- \int_\Omega p\xi_2^{p-1} (v - \underline{v})_- \right) \leq$$

$$c_2(\xi_2, \Omega) \|\underline{U}_-\|_{L^1(\Omega)} \|\underline{V}_-\|_{L^1(\Omega)} \leq c_2(\xi_2, \Omega) \left(\|\underline{U}_-\|^2_{L^2(\Omega)} + \|\underline{V}_-\|^2_{L^2(\Omega)} \right)$$

and

$$\int_\Omega \left(\overline{V}_+ \int_\Omega (u^q - \overline{u}^q) \right) \leq c_3(\xi_3, \Omega) \left(\|\overline{U}_+\|^2_{L^2(\Omega)} + \|\overline{V}_+\|^2_{L^2(\Omega)} \right),$$

$$-\int_\Omega \left(\underline{V}_- \int_\Omega (u^q - \underline{u}^q) \right) \leq c_4(\xi_4, \Omega) \left(\|\underline{U}_-\|^2_{L^2(\Omega)} + \|\underline{V}_-\|^2_{L^2(\Omega)} \right).$$

The rest of the proof follows the proof of Theorem 2.1. □

We now consider different cases, depending on the values of χ_i. For completeness of the proof we also include the non-chemotactic case $\chi_1 = \chi_2 = 0$.

Case I. $\chi_1 = \chi_2 = 0$.

In that case the system (57) and (59) contains two positive steady states if

$$p + q > 2 \tag{62}$$

defined by

$$(0, 0), \qquad \text{and} \qquad (u^*, v^*) := \left(a^{\frac{1}{pq-1}} b^{\frac{p}{pq-1}}, a^{\frac{q}{pq-1}} b^{\frac{1}{pq-1}} \right).$$

Lemma 3.5. *We have the following*

a.- If $p = q = 1$ and $ab > 1$, any solution satisfies

$$\lim_{t\to\infty} (\overline{u}, \overline{v}) = (0, 0).$$

b.- If $p \geq 1$ and $q \geq 1$, under assumption (62):

b_1.- There exists a solution $(\overline{u}_1, \overline{v}_1)$ such that

$$\begin{gathered} \lim_{t\to\infty} (\overline{u}_1, \overline{v}_1) = (0, 0), \ \lim_{t\to-\infty} (\overline{u}_1, \overline{v}_1) = (u^*, v^*), \\ \overline{u}_1' < 0 \ \text{ and } \ \overline{v}_1' < 0 \ \text{ for } \ t \in I\!R. \end{gathered} \tag{63}$$

b_2.- *There exists a solution* $(\overline{u}_2, \overline{v}_2)$ *such that*

$$\begin{gathered} \lim_{t\to-\infty}(\overline{u}_2, \overline{v}_2) = (u^*, v^*), \quad \lim_{t\to T}(\overline{u}_2, \overline{v}_2) = (\infty, \infty), \\ u_2' > 0 \ \text{ and } v_2' > 0 \ \text{ for } \ t < T, \end{gathered} \tag{64}$$

for some $T \leq \infty$.

Proof. For the case (a) we have a linear system where $(0,0)$ is the steady state and both eigenvalues are negative as a consequence of $ab > 1$. Therefore $(0,0)$ is asymptotically stable which proves (a).

For the second case there exists two steady states $(0,0)$ and (u^*, v^*). We consider the regions

$$A_1 := \{(\overline{u}, \overline{v}) \in I\!R^2 \text{ such that } -a\overline{u} + \overline{v}^p \leq 0, \ -b\overline{v} + \overline{u}^q \leq 0, \ \overline{u} \leq u^* \text{ and } \overline{v} \leq v^*\},$$

$$A_2 := \{(\overline{u}, \overline{v}) \in I\!R^2 \text{ such that } -a\overline{u} + \overline{v}^p \geq 0, \ -b\overline{v} + \overline{u}^q \geq 0, \ \overline{u} \geq u^* \text{ and } \overline{v} \geq v^*\}.$$

Notice that

- A_1 contains both steady states,
- $\overline{u}' < 0$ and $\overline{v}' < 0$ in the interior of A_1,
- $(\overline{u}', \overline{v}') \cdot \nu < 0$ in ∂A_1 except for $(0,0)$ and (u^*, v^*). Where ν is the exterior unit normal vector to A_1.

Since there is not any steady state or periodic solution in the interior of A_1, we have that there exists a solution $(\overline{u}_1, \overline{v}_1)$ satisfying (63).

In the same way, thanks to $(\overline{u}', \overline{v}') \cdot \nu < 0$ in ∂A_2 (except in the critical point (u^*, v^*)) and $\overline{u}' > 0$ and $\overline{v}' > 0$ in the interior of A_2 we obtain that there exists a solution $(\overline{u}_2, \overline{v}_2)$ such that

$$\lim_{t\to-\infty}(\overline{u}_2, \overline{v}_2) = (u^*, v^*), \quad \lim_{t\to T} |\overline{u}_2| + |\overline{v}_2| = \infty, \tag{65}$$

for some $T \leq \infty$. If

$$\lim_{t\to T} \overline{u}_2(t) < \infty \quad \text{or} \quad \lim_{t\to T} \overline{v}_2(t) < \infty$$

occurs, we have that $\overline{u}_2$ and $\overline{v}_2$ are uniformly bounded in $(0,T)$ which contradicts (65) and proves (64). □

Case II. $\chi_1 + \chi_2 > 0$

Lemma 3.6. *Let* $p, q \geq 1$ *and* $a, b > 0$, *such that there exists* $s^* \leq 1$ *and*

$$-\min\{a,b\}s + s^{\min\{p,q\}} + 2\max\{\chi_1, \chi_2\}s^2 < 0 \tag{66}$$

for any $s < s^*$, *we have that for* $0 \leq \overline{u}_0 + \overline{v}_0 < s^*$ *the solution* $(\overline{u}, \overline{v})$ *to the problem satisfies*

$$(\overline{u}, \overline{v}) \to (0,0) \qquad \text{as } \ t \to \infty.$$

Proof. Notice that, since the solutions are positive, the solutions $(\overline{u}_1, \overline{v}_1)$ to the following system are upper bounds of the solutions to (57)-(60).

$$\overline{u}_1' = -a\overline{u}_1 + \overline{v}_1^p + \chi_1 \overline{u}_1(\overline{u}_1 + \overline{v}_1), \quad t > 0, \tag{67}$$

$$\overline{v}_1' = -b\overline{v}_1 + \overline{u}_1^q + \chi_2 \overline{v}_1(\overline{u}_1 + \overline{v}_1), \quad t > 0. \tag{68}$$

We add the above expressions to obtain

$$\overline{u}_1' + \overline{v}_1' \leq -\min\{a,b\}(\overline{u}_1 + \overline{v}_1) + (\overline{u}_1 + \overline{v}_1)^{\min\{p,q\}} + 2\max\{\chi_1, \chi_2\}(\overline{u}_1 + \overline{v}_1)^2,$$

since the solution w to the equation

$$w' = -\min\{a,b\}w + w^{\min\{p,q\}} + 2\max\{\chi_1,\chi_2\}w^2,$$
$$0 < w(0) < s^*$$

satisfies

$$\lim_{t\to\infty} w = 0$$

we have, by Maximum Principle that

$$\lim_{t\to\infty} \overline{u}_1 = \lim_{t\to\infty} \overline{v}_1 = 0$$

and therefore

$$\lim_{t\to\infty} \overline{u} = \lim_{t\to\infty} \overline{v} = 0$$

if

$$\overline{u}_0 + \overline{v}_0 < s^*.$$

□

Now we formulate a comparison result similar to Theorem 2.1:

Theorem 3.7. • *Case I,* $\chi_1 = \chi_2 = 0$

i- If $p = q = 1$ *and* $ab > 1$ *the solution to (54)-(55) satisfies*

$$\lim_{t\to\infty} \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} = 0$$

for any positive and bounded initial data.

ii_1- If $p \geq 1$ *and* $q \geq 1$ *and (62) is satisfied, we have that the solution to (54)-(55) satisfies*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \longrightarrow 0 \quad \text{as } t \to \infty,$$

if the non negative initial data satisfy

$$\sup_\Omega u_0 < u^* \quad \text{and} \quad \sup_\Omega v_0 < v^*. \tag{69}$$

ii_2- If $p \geq 1$ *and* $q \geq 1$ *and (62) is satisfied, then, the solution to (54)-(55) satisfies*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \longrightarrow \infty \quad \text{as } t \to T,$$

for some $T \leq \infty$*, if the initial data satisfy*

$$\inf_\Omega u_0 > u^* \quad \text{and} \quad \inf_\Omega v_0 > v^*. \tag{70}$$

• *Case II.* $\chi_1 + \chi_2 > 0$ *if either*

II_i- If $p, q > 1$, $a, b > 0$

or

II_{ii}- If $\min\{p,q\} = 1$ *and* $\min\{a,b\} > 1$,

then, for any initial data (u_0, v_0) *satisfying*

$$\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} < s^*$$

(for s^* *defined in Lemma 3.6) we have that*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \longrightarrow 0 \quad \text{as } t \to \infty.$$

Proof. The proof of "i" is a direct consequence of Theorem 3.4 and Lemma 3.5-(a) provided the initial data $(\overline{u}_0, \overline{v}_0)$ satisfy

$$\underline{u}_0 < u_0 < \overline{u}_0, \quad \underline{v}_0 < v_0 < \overline{u}_0.$$

For the case ii_1, the assumption (69) implies that there exists bounded initial data $(\overline{u}_0, \overline{v}_0)$ which belong to the trajectory $(\overline{u}_1, \overline{v}_1)$ and satisfy $u_0 \leq \overline{u}_0$ and $v_0 \leq \overline{v}_0$. Thanks to Theorem 3.4 and Lemma 3.5-(b_1) we conclude ii_1.

If (70) is verified, we take initial data $(\underline{u}_0, \underline{v}_0)$ in the trajectory $(\overline{u}_2, \overline{v}_2)$ such that $u_0 \geq \underline{u}_0$, and $v_0 \geq \underline{v}_0$. We apply Theorem 3.4 and Lemma 3.5-(b_2) to obtain ii_2.

The proof of case II is based in a comparison argument for the system of ODEs. We consider initial data $(\overline{u}_0, \underline{u}_0, \overline{v}_0, \underline{v}_0)$ such that

$$\overline{u}_0 + \overline{v}_0 < s^*,$$

for s^* defined in Lemma 3.6. Since assumption (66) is satisfied for the cases II_i and II_{ii} we apply Lemma 3.6 and Theorem 3.4 to end the proof. □

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E-mail address: negreanu@mat.ucm.es
E-mail address: jtello@eui.upm.es