Differential and Integral Equations Volume 29, Numbers 5-6 (2016), 441–454

MATHEMATICAL ANALYSIS OF A MODEL OF CHEMOTAXIS WITH COMPETITION TERMS

Akisato Kubo
School of Health Sciences, Fujita Health University
1-98 Toyoake, Aichi, Japan

J. Ignacio Tello
Depto. Matemática Aplicada T.I.C. E.T.S.I. Sistemas Informáticos
Universidad Politécnica de Madrid, 28031 Madrid, Spain

(Submitted by: Takashi Suzuki)

Abstract. We consider a competitive system of differential equations describing the behavior of two biological species "u" and "v". The system is weakly coupled and one of the species has the capacity to diffuse and moves toward the higher concentration of the second species following its gradient, the density function satisfies a second order parabolic equation with chemotactic terms. The second species does not have motility capacity and satisfies an ordinary differential equation. We prove that the solutions are uniformly bounded and exist globally in time. The asymptotic behavior of solutions is also studied for a range of parameters and initial data. If the chemotaxis coefficient χ is small enough the quadratic terms drive the solutions to the constant steady state.

1. Introduction

Predator-Prey systems have been studied from the beginning of the 20th century when Lotka and Volterra proposed a system of Ordinary Differential Equations to describe the evolution of the total population of predators and preys. The system is also generalized as cooperative and competitive systems depending on the signs of the parameters. Later, linear diffusion is included in the equations to describe the random motility of the individuals. The model consists of a system of second order PDEs of parabolic type with reaction diffusion terms, see for instance Cosner and Lazer [4] for details.

In the last decades, chemotaxis, hapotaxis and prey-taxis have been introduced in the mathematical models of differential equations by second order

AMS Subject Classifications: 35B40, 35K57, 92D40.
Accepted for publication: November 2015.

terms. We consider a system where one of the species, denoted by "v" does not have capacity of movement, while the other species, denoted by "u", diffuses and moves towards the population "v".

The densities of populations u and v are considered in a bounded domain $\Omega \subset I\!R^N$ with a regular boundary. The system is presented in terms of partial differential in the following form:

$$\frac{\partial u}{\partial t} - \Delta u = -\text{div}\,(\chi u \nabla v) + \mu_1 u(1 - u - a_1 v), \qquad t > 0, \quad x \in \Omega \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \mu_2 v(1 - a_2 u - v), \quad t > 0 \tag{1.2}$$

with a boundary condition

$$\frac{\partial u}{\partial n} - \chi u \frac{\partial v}{\partial n} = 0, \tag{1.3}$$

and initial data

$$u(x,0) = u_0(x); \qquad v(x,0) = v_0(x), \qquad x \in \Omega \tag{1.4}$$

where μ_1 and μ_2 are positive constants. Notice that:

- The case where $a_1 > 0$ and $a_2 > 0$ is a competitive system where the population u has taxis abilities and diffuses.
- The case where $a_1 < 0$ and $a_2 < 0$ is a cooperative system.
- The case where $a_1 < 0$ and $a_2 > 0$ is a prey-taxis system, where the predator has the ability to orientate its movement towards a higher concentration of prey.
- The case where $a_1 > 0$ and $a_2 < 0$ is a predator-prey system, where the predators do not have the ability to move and the prey orientates its movement following a predator gradient, if $\chi < 0$ moving to a lower concentration of predators and for $\chi > 0$ towards the higher concentration.

One of the main issues in population dynamics is to obtain the range of parameters and initial data which drives the populations to coexistence or to extinction. The case without taxis has been well studied in the literature and the behavior of the populations is given by the parameters a_1 and a_2 (see for instance [4] and references therein). Here, we also consider the taxis of one of the species. Systems with competitive terms and taxis have been already studied for two or more populations, see for instance [1], [3], [11], [14], [19], [20] and references therein.

Parabolic-ODE systems with chemotactic terms have been used to model biological processes in last decades, see for instance [5], [9], [10], [15], [16], [17], and references therein. Global existence of solutions and stability of steady states for Parabolic-ODE chemotactic systems have been also studied, see for instance [6], [13] and [18]. In [12], numerical simulations suggest that for a range of initial data and reaction terms blow up of solutions occurs.

In this work we present results on global existence of solutions and coexistence of the populations for "weak" coupled conditions, i.e.

$$|a_i| < 1, \qquad \text{for} \;\; i = 1, 2 \tag{1.5}$$

where $\chi > 0$ (positive taxis). The article is organized as follows: In section 2 we study the global-in-time existence of solutions by using a priori estimates in L^∞ and abstract theory of PDEs (see Amman [2]). In section 3 some extra assumptions are required to obtain the coexistence, we also prove that the homogeneous steady state

$$u^* = \frac{1-a_1}{1-a_1a_2}, \quad v^* = \frac{1-a_2}{1-a_1a_2}$$

is asymptotically stable (see Theorem 3.1).

To prove the global boundedness of solutions we assume that

$$u_0,\ v_0 \geq 0, \qquad (u_0, v_0) \in (W^{1,s}(\Omega))^2, \quad \text{for some } \ s > \max\{4, N\}. \tag{1.6}$$

Since $W^{1,s}(\Omega)) \subset L^\infty(\Omega)$ we have that u_0 and v_0 are uniformly bounded. We introduce

$$\overline{u} := \|u_0\|_{L^\infty(\Omega)}, \qquad \overline{v} = \max\{1, \|v_0\|_{L^\infty(\Omega)}\} \tag{1.7}$$

then

$$0 \leq u_0 \leq \overline{u}, \tag{1.8}$$

$$0 \leq v_0 \leq \overline{v}. \tag{1.9}$$

Throughout the article we also assume that

$$\min_{0\leq v\leq \overline{v}} \{\mu_1 - \chi a_2\mu_2 v\} > 0. \tag{1.10}$$

Notice that if $v_0 \leq 1$, assumption (1.10) is satisfied if

$$\mu_1 > \chi\mu_2 a_2.$$

We now define the function h by

$$h(v) := \frac{e^{-\chi v}(\mu_1 - (\mu_1 a_1 + \chi\mu_2)v + \chi\mu_2 v^2)}{\mu_1 - \chi\mu_2 a_2 v} \tag{1.11}$$

and denote by $\overline{h}$ the following upper bound of h,

$$\overline{h} := \max_{0\leq v\leq \overline{v}} h(v). \tag{1.12}$$

Notice that

$$1 = h(0) \leq \overline{h} \leq \frac{\mu_1(1+|a_1|\overline{v}) + \chi\mu_2\overline{v}^2}{\min\limits_{0\leq v\leq \overline{v}}\{\mu_1 - \chi\mu_2 a_2 v\}}.$$

Analogously, we define $\underline{h}$ by the minimum of h,

$$\underline{h} := \min_{0\leq v\leq \overline{v}} h(v). \tag{1.13}$$

Let us consider

$$\overline{h}' := \max_{0\leq v\leq 1} |h'(v)|. \tag{1.14}$$

Notice that

$$h' = -\chi h(v) + \frac{e^{-\chi v}(-\mu_1 a_1 - \chi\mu_2 + 2\chi\mu_2 v) + \chi\mu_2 a_2 h(v)}{\mu_1 - \chi\mu_2 a_2 v}$$

and

$$0 < \overline{h}' \leq \chi\overline{h} + \frac{\chi\mu_2(2\overline{v}-1) + \mu_1(a_1)_+ + \chi\mu_2(a_2)_+\overline{h}}{\min\limits_{0\leq v\leq \overline{v}}\{\mu_1 - \chi\mu_2 a_2 v\}}$$

where $(s)_+$ is the positive part function defined by

$$(s)_+ := \begin{cases} s & \text{if } s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In order to study the asymptotic behavior, some extra assumptions are required. We define $\overline{w}$ as follows

$$\overline{w} := \max\{\|u_0 e^{-\chi v_0}\|_{L^\infty(\Omega)}, \overline{h}\} \tag{1.15}$$

and assume

$$\overline{w} < \frac{e^{-\chi}}{a_2} \tag{1.16}$$

and

$$1 \geq v_0 \geq \inf_{x\in\Omega}\{v_0\} > 0. \tag{1.17}$$

We define $\underline{v}$ and $\underline{w}$ by

$$\underline{v} := \min\{\inf_{x\in\Omega}\{v_0\}, 1 - a_2\overline{w}e^{\chi}\} > 0, \tag{1.18}$$

$$\underline{w} := \min\{\inf_{x\in\Omega}\{u_0 e^{-\chi v_0}\}, \underline{h}\} > 0. \tag{1.19}$$

We also assume

$$\mu_1 > \chi\mu_2 \tag{1.20}$$

$$4\mu_1\underline{w} > \chi\mu_2 \tag{1.21}$$

and finally

$$\frac{(\underline{w}\mu_1 - \frac{\chi\mu_2}{4})\underline{v}\mu_2(1 + 2\chi w^* e^{-\chi(1-v^*)})}{\mu_2\overline{h}'^2 e^{2\chi}\overline{w}(\mu_1 - \chi\mu_2 a_2\underline{v})} > a_2^2 \tag{1.22}$$

Notice that if $a_2 \to 0$ or $\chi \to 0$ we have that assumptions (1.16) and (1.22) are satisfied. For simplicity we assume that $|\Omega| = 1$.

2 Global existence of solutions

In order to study the global-in-time existence of solutions we start with the following lemma.

Lemma 2.1 *Under assumptions (1.5)-(1.10) there exists* $T_{max} > 0$ *and a unique maximal solution to (1.1)–(1.4) in* $(0, T_{max})$ *satisfying*

$$(u,v) \in C([0,T_{max}), (W^{1,s}(\Omega))^2) \cap C^1((0,T_{max}), ((W^{1,s}(\Omega))') \times W^{1,s}(\Omega)) \tag{2.23}$$

and

$$\limsup_{t\to T_{max}} \left(\|u(t)\|_{W^{1,s}(\Omega)} + \|v(t)\|_{W^{1,s}(\Omega)} + t\right) = \infty. \tag{2.24}$$

Proof: We introduce the following notation

$$\mathcal{A}_1 u = -\Delta u; \qquad \mathcal{A}_2(u)v = \mathrm{div}(u\chi\nabla v)$$

and

$$\mathcal{B}_1 u = \frac{\partial u}{\partial n}, \qquad \mathcal{B}_2(u)v = \chi u \frac{\partial v}{\partial n},$$

then, the system (1.1)-(1.4) can be expressed as follows

$$\begin{cases} u_t + \mathcal{A}_1 u + \mathcal{A}_2(u)v = g_1(u,v) & \text{in } \Omega \times (0, T_{max}), \\ v_t = g_2(u,v) & \text{in } \Omega \times (0, +\infty), \\ \mathcal{B}_1 u + \mathcal{B}_2(u)v = 0 & \text{on } \partial\Omega \times (0, T_{max}), \end{cases}$$

where $g_1(u,v) = \mu_1 u(1 - u - a_1 v)$ and $g_2(u,v) = \mu_2 v(1 - a_2 u - v)$.

We apply Theorem 6.4 in [2] to obtain the existence of a unique maximal weak solution satisfying (2.23), (2.24). □

To continue with the proof of the global-in-time existence, we introduce the following change of unknown:

$$u = we^{\chi v} \tag{2.25}$$

to obtain

$$e^{\chi v}\frac{\partial}{\partial t}w - \nabla(e^{\chi v}\nabla w) = \mu_1 w e^{\chi v}(1 - we^{\chi v} - a_1 v) - \chi e^{\chi v} w \mu_2 v[1 - a_2 w e^{\chi v} - v], \tag{2.26}$$

$$\frac{\partial v}{\partial t} = \mu_2 v(1 - a_2 w e^{\chi v} - v). \tag{2.27}$$

Notice that the main difficulty in the problem is the non-linear second order term

$$div(\chi u \nabla v). \tag{2.28}$$

Since v satisfies an O.D.E. by introducing the change of unknown (2.25), the second order term (2.28) becomes a nonlinear first order term plus a reaction quadratic term.

We denote by f the function

$$f(w,v) := e^{\chi v}\left(\mu_1 - \chi\mu_2 a_2 v\right)\left[\frac{e^{-\chi v}(\mu_1 - \mu_1 a_1 v - \chi\mu_2 v + \chi\mu_2 v^2)}{\mu_1 - \chi\mu_2 a_2 v} - w\right]$$

then, equation (2.26) can be expressed in the following two fashions:

$$e^{\chi v}\frac{\partial}{\partial t}w - \nabla(e^{\chi v}\nabla w) = we^{\chi v} f(w,v), \tag{2.29}$$

$$\frac{\partial w}{\partial t} - \Delta w - \chi \nabla w \nabla v = w f(w,v). \tag{2.30}$$

Notice that f is bounded in $(0, T_{\max} - \epsilon)$ for any $\epsilon > 0$.

Remark 2.1 *Since*

$$(w,v) \in C([0,T_{max}),(W^{1,s}(\Omega))^2) \cap C^1((0,T_{max}),((W^{1,s}(\Omega))') \times W^{1,s}(\Omega))$$

and $s > \max\{4,N\}$, *we have that*

$$(w,v) \in C(0,T_{\max} : L^\infty(\Omega))^2$$

which gives

$$w_t - \Delta w \in C(0,T_{\max} : L^2(\Omega)).$$

It implies that

$$w \in C(0,T_{\max} : H^2(\Omega)),$$

and $v \in C(0,T_{\max} : H^2(\Omega))$.

Remark 2.2 *We notice that the solution* (w,v) *exists in the interval* $(0,T_{max})$ *and*

$$(w,v) \in C([0,T_{max}),(W^{1,s}(\Omega))^2) \cap C^1((0,T_{max}),((W^{1,s}(\Omega))') \times W^{1,s}(\Omega)).$$

Lemma 2.2 *Under assumptions (1.5)-(1.10) we have that*

$$w \geq 0, \qquad v \geq 0.$$

Proof: We multiply by $-(-v)_+$ both sides of (2.27), since w is bounded in $(0,T_{max}-\epsilon)$ we have that

$$\frac{d}{dt}(-v)_+^2 = a(x,t)(-v)_+^2, \qquad t \in (0,T_{max}-\epsilon)$$

for a bounded function $a(x,t)$. Thanks to nonnegativity of v_0 and Gronwall's Lemma we obtain $(-v)_+ = 0$ and proves $v \geq 0$ in $(0,T_{max}-\epsilon)$. We take limits when $\epsilon \to 0$ to obtain

$$v \geq 0 \;\text{ in } (0,T_{max}). \tag{2.31}$$

In the same way we multiply by $-(-w)_+$ the both sides of (2.29) and integrate over Ω to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(-w)_+^2 + \int_\Omega e^{\chi v}|\nabla(-w)_+|^2 = \int_\Omega (-w)_+^2 e^{\chi v}[f(w,v) + \frac{\chi}{2}\mu_2 v(1-v-a_2 w e^{\chi v})].$$

Since $f(w,v)$ and $\frac{\chi}{2}\mu_2 v(1-v-a_2 w e^{\chi v})$ are bounded in $[0,T_{\max}-\epsilon)$, for any $\epsilon > 0$, we have that

$$\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(-w)_+^2 \leq A\int_\Omega e^{\chi v}(-w)_+^2,$$

for some $A > 0$. Gronwall's Lemma gives

$$(-w)_+ = 0, \qquad t \in [0,T_{max}-\epsilon).$$

We now take limits when $\epsilon \to 0$ and thanks to (2.31) the proof ends. □

Lemma 2.3 *Let $\overline{v}$ and $\overline{w}$ as defined in (1.7) and (1.15), i.e.*

$$\overline{w} := \max\{\overline{h}, \|w_0\|_{L^\infty(\Omega)}\}, \qquad \overline{v} = \max\{1, \sup\{v_0(x)\}\}$$

then, under assumptions (1.5)-(1.10) we have that

$$u \leq \overline{w}e^{\chi\overline{v}}, \qquad v \leq \overline{v}.$$

Proof: Thanks to Lemma 2.2 we have that $v, w \geq 0$ and therefore $u = we^{\chi v} \geq 0$, then, the equation for v gives

$$v_t \leq \mu_2 v(1 - v)$$

and maximum principle proves that

$$v \leq \max\{1, \sup\{v_0(x)\}\} = \overline{v}. \tag{2.32}$$

Let $\overline{h}$ and $\overline{w}$ be defined by (1.12) and (1.15) respectively. We multiply the both sides of equation (2.29) by $(w - \overline{w})_+$ to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(w - \overline{w})_+^2 + \int_\Omega e^{\chi v}|\nabla(w - \overline{w})_+|^2$$

$$= \int_\Omega (w - \overline{w})_+ e^{\chi v}[wf(w, v) + (\mu_2 v(1 - v - a_1 w e^{\chi v})\frac{\chi}{2}(w - \overline{w})_+]$$

$$\leq \int_\Omega (w - \overline{w})_+ e^{\chi v}[wf(w, v) + \mu_2 v\frac{\chi}{2}(w - \overline{w})_+].$$

Since the term

$$(w - \overline{w})_+ e^{2\chi v} wf(w, v) = (w - \overline{w})_+ e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)\,[h(v) - w]$$

by definition of $\overline{h}$ and $\overline{w}$ (see (1.12 and (1.15)), we have

$$h(v) \leq \overline{h} \leq \overline{w},$$

then

$$(w - \overline{w})_+ e^{2\chi v} wf(w, v) \leq 0$$

and it results

$$\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(w - \overline{w})_+^2 \leq \frac{\chi\mu_2}{2}\overline{v}\int_\Omega e^{\chi v}(w - \overline{w})_+^2.$$

Thanks to Gronwall's Lemma we prove that

$$w \leq \overline{w}$$

which ends the proof. □

Theorem 2.1 *Under assumptions (1.5)-(1.10), there exists a unique solution to (1.1)-(1.4) for $T_{max} = \infty$ such that*

$$(w, v) \in C([0, \infty), (W^{1,s}(\Omega))^2) \cap C^1((0, \infty), ((W^{1,s}(\Omega))') \times W^{1,s}(\Omega)).$$

The proof is a consequence of previous lemmas. □

3 Asymptotic behavior

Notice that, since $a_i \in (-1,1)$ for $i = 1,2$ we have that the unique positive homogeneous steady states are given by

$$u^* = \frac{1-a_1}{1-a_1a_2}, \quad v^* = \frac{1-a_2}{1-a_1a_2}.$$

In this section we study the asymptotic stability of the positive homogenous steady states. The result is enclosed in the following theorem:

Theorem 3.1 *Under assumptions of Theorem 2.1 and (1.16)-(1.22) the unique solution to problem (1.1)-(1.4) satisfies*

$$\int_\Omega (u-u^*)^2 + \int_\Omega (v-v^*)^2 \leq c_0 e^{-\epsilon t}$$

for some $\epsilon > 0$ and $c_0 > 0$.

The proof of the theorem is divided into several steps, for reader's convenience we present them in 5 separate lemmata.

Lemma 3.1 *Under assumptions (1.5)-(1.10), (1.16)-(1.22) there exists $\underline{w} > 0$ such that*

$$u \geq \underline{w}.$$

Proof: The proof is similar to the proof of Lemma 2.3. Let us consider $\underline{w}$ defined in (1.19). Notice that to guarantee the positivity of $\underline{w}$ we only have to prove that $\mu_1 - (\mu_1 a_1 + \chi\mu_2)v + \chi\mu_2 v^2 > 0$ in $[0,1]$. We first notice that $p(x) := ax^2 - 2bx + c$ attains its minimum at $x = b/a$ (for $a > 0$) and

$$\min_{x\in I\!R}\{p(x)\} = p(b/a) = c - b^2/a.$$

Thanks to assumption (1.20) we have that $\mu_1 > \mu_2\chi$. We now consider three different cases:

- Case 1: $a_1 \leq 0$. We have that

$$\begin{aligned}
\min_{0\leq v\leq 1}\{h(v)\} &= \min_{0\leq v\leq 1}\left\{\frac{e^{-\chi v}(\mu_1 - (\mu_1 a_1 + \chi\mu_2)v + \chi\mu_2 v^2)}{\mu_1 - \chi\mu_2 a_2 v}\right\} \\
&\geq \frac{1}{\mu_1 + \chi\mu_2|a_2|}\min_{0\leq v\leq 1}\{(\mu_1 - \chi\mu_2 v)\} \\
&\geq \frac{1}{\mu_1 + \chi\mu_2|a_2|}(\mu_1 - \chi\mu_2)) > 0.
\end{aligned}$$

- Case 2: $a_1 > 0$ and $\mu_1 a_1 < \chi\mu_2$.

$$\begin{aligned}
\min_{0\leq v\leq 1}\{h(v)\} &= \min_{0\leq v\leq 1}\left\{\frac{e^{-\chi v}(\mu_1 - (\mu_1 a_1 + \chi\mu_2)v + \chi\mu_2 v^2)}{\mu_1 - \chi\mu_2 a_2 v}\right\} \\
&\geq \frac{1}{\mu_1 + \chi\mu_2|a_2|}\min_{0\leq v\leq 1}\{(\mu_1 - (\mu_1 a_1 + \chi\mu_2)v + \chi\mu_2 v^2)\} \\
&\geq \frac{1}{\mu_1 + \chi\mu_2|a_2|}\left(\mu_1 - \frac{(\mu_1 a_1 + \chi\mu_2)^2}{4\chi\mu_2}\right) \\
&\geq \frac{1}{\mu_1 + \chi\mu_2|a_2|}(\mu_1 - \chi\mu_2)) > 0.
\end{aligned}$$

- Case 3: $a_1 > 0$ and $\mu_1 a_1 \geq \chi\mu_2$. Then,

$$\begin{aligned}\min_{0\leq v\leq 1}\{h(v)\} &\geq e^{-\chi}\tfrac{1}{\mu_1+\chi\mu_2|a_2|}\max_{0\leq v\leq 1}(\mu_1-(a_1\mu_1+\chi\mu_2)v+\chi\mu_2 v^2)\\ &> \tfrac{e^{-\chi}}{\mu_1+\chi\mu_2|a_2|}\mu_1(1-a_1)>0.\end{aligned}$$

We multiply the both sides of equation (2.29) by $-(\underline{w}-w)_+$ to have

$$\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(\underline{w}-w)_+^2+\int_\Omega e^{\chi v}|\nabla(\underline{w}-w)_+|^2$$

$$=-\int_\Omega(\underline{w}-w)_+e^{\chi v}[wf(w,v)+(\mu_2 v(1-v-a_1we^{\chi v})\frac{\chi}{2}(\underline{w}-w)_+]$$

since

$$-(\underline{w}-w)_+e^{\chi v}wf(w,v)=-(\underline{w}-w)_+e^{\chi v}w(\mu_1-\chi\mu_2a_2v)\,[h(v)-w]\leq 0$$

we apply Gronwall's lemma to obtain $w\geq\underline{w}$. The proof ends thanks to the positivity of v. □

Lemma 3.2 *Under assumptions of theorem 3.1, there exists $\underline{v}>0$ such that*

$$v\geq\underline{v}:=\min\left\{\inf_{x\in\Omega}\{v_0\},1-a_2\overline{w}e^{\chi}\right\}>0.$$

Proof: First, we notice that thanks to assumption (1.16), we know that $1-a_2\overline{w}e^{\chi}>0$ and therefore $\underline{v}>0$. We multiply equation (1.2) by $-(\underline{v}-v)_+$ to get

$$\frac{d}{dt}\frac{1}{2}(\underline{v}-v)_+^2=-\mu_2 v(\underline{v}-v)_+(1-a_2u-v)$$

since $u\leq\overline{w}e^{\chi}$ we have

$$-v(\underline{v}-v)_+(1-a_2u-v)\leq -v(\underline{v}-v)_+(1-a_2\overline{w}e^{\chi}-v).$$

From $1-a_2\overline{w}e^{\chi}\geq\underline{v}>0$ it follows that $1-a_2\overline{w}e^{\chi}-v>0$ if $v\leq\underline{v}$, which gives

$$-v(\underline{v}-v)_+(1-a_2\overline{w}e^{\chi}-v)\leq -v(\underline{v}-v)_+(\underline{v}-v)\leq 0.$$

Then

$$\frac{d}{dt}\frac{1}{2}(\underline{v}-v)_+^2\leq 0$$

which ends the proof. □

Lemma 3.3 *Under assumptions (1.5)-(1.10), (1.16)-(1.22) we have*

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(w-w^*)^2+\int_\Omega e^{\chi v}|\nabla(w-w^*)|^2\leq\\ &-\frac{1}{2}\int_\Omega(w-w^*)^2e^{2\chi v}w(\mu_1-\frac{\chi\mu_2}{4\underline{w}})+\frac{\overline{h}'^2}{2}\int_\Omega|v-v^*|^2e^{2\chi v}w(\mu_1-\chi\mu_2a_2v).\end{aligned}\tag{3.33}$$

Proof: We define w^* by

$$w^* := u^* e^{-\chi v^*},$$

then, we multiply the both sides of equation (2.29) by $(w - w^*)$ to obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(w-w^*)^2 + \int_\Omega e^{\chi v}|\nabla(w-w^*)|^2 \\ &= \int_\Omega (w-w^*)e^{2\chi v} w f(w,v) + \frac{\chi\mu_2}{2}\int_\Omega v(1-v-a_2 w e^{\chi v})(w-w^*)^2 e^{\chi v}. \end{aligned} \tag{3.34}$$

We have

$$(w-w^*)e^{2\chi v} w f(w,v) = (w-w^*)e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)\left[h(v) - w\right],$$

for h defined by (1.11). Notice that $h(v^*) = w^*$ and

$$h(v) - w = (w^* - w) + h(v) - h(v^*) = (w - w^*) + h'(s)(v - v^*)$$

for some $s \in (\underline{v}, 1)$. Then

$$(w-w^*)e^{2\chi v} w f(w,v) = (w-w^*)e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)[(w^* - w) + h'(s)(v - v^*)]$$

and the last term in the previous equation

$$(w-w^*)e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)[h'(s)(v-v^*)] \le \overline{h}'|w-w^*||v-v^*|e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)$$

thanks to Young's inequality

$$\le \frac{|\overline{h}'|^2}{2}|v-v^*|^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v) + \frac{1}{2}|w-w^*|^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v).$$

Since $v(1-v)$ attains its maximum at $v = 1/2$, we have the following inequality

$$\frac{\chi\mu_2}{2}\int_\Omega v(1-v-a_2 w e^{\chi v})(w-w^*)^2 e^{\chi v} \le \frac{\chi\mu_2}{2}\int_\Omega (\frac{1}{4} - a_2 w v e^{\chi v})(w-w^*)^2 e^{\chi v}.$$

Thanks to assumption (1.10), (3.34) becomes

$$\frac{d}{dt}\frac{1}{2}\int_\Omega e^{\chi v}(w-w^*)^2 + \int_\Omega e^{\chi v}|\nabla(w-w^*)|^2 \le -\frac{1}{2}\int_\Omega (w-w^*)^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)$$

$$+\frac{\chi\mu_2}{2}\int_\Omega (\frac{1}{4} - a_2 w v e^{\chi v})(w-w^*)^2 e^{\chi v} + \frac{\overline{h}'^2}{2}\int_\Omega |v-v^*|^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v) \le$$

$$-\frac{1}{2}\int_\Omega (w-w^*)^2 e^{2\chi v} w(\mu_1 - \frac{\chi\mu_2}{4\underline{w}}) + \frac{\overline{h}'^2}{2}\int_\Omega |v-v^*|^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)$$

and the proof of the lemma ends. □

Lemma 3.4 *Under assumptions (1.5)-(1.10), (1.16)-(1.22) we have the following inequality*

$$\frac{\partial}{\partial t}\int_\Omega (v-v^*)^2 \le -\mu_2(1 + 2\chi w^* e^{-\chi(1-v^*)})\int_\Omega v(v-v^*)^2 + \mu_2 a_2^2 \int_\Omega e^{2\chi v} v(w^* - w)^2. \tag{3.35}$$

Proof: Since $1 - v - a_2 u = v^* - v + a_2(u^* - u)$, we have that, after multiplying the both sides of (1.2) by $(v - v^*)$

$$\frac{\partial}{\partial t}\frac{1}{2}(v - v^*)^2 = -\mu_2 v(v - v^*)^2 + \mu_2 a_2 v(u^* - u)(v - v^*).$$

Notice that

$$\begin{aligned} u^* - u &= e^{\chi v}(w^* e^{\chi(v^*-v)} - w) \\ &= e^{\chi v}(w^* - w + w^*(e^{\chi(v^*-v)} - 1)) \\ &= e^{\chi v}(w^* - w - \chi w^*(e^{\chi(v^*-s)}(v - v^*)) \end{aligned}$$

for some $s \in (\underline{v}, \overline{v})$.
Since

$$\frac{\partial}{\partial t}\frac{1}{2}(v - v^*)^2 = -\mu_2 v(v - v^*)^2(1 + \chi w^* e^{\chi(v^*-s)}) + \mu_2 a_2 v(v - v^*)e^{\chi v}(w^* - w),$$

$$\frac{\partial}{\partial t}\frac{1}{2}(v - v^*)^2 \leq -\mu_2 v(v - v^*)^2(1 + \chi w^* e^{-\chi(1-v^*)}) + \mu_2 a_2 v(v - v^*)e^{\chi v}(w^* - w)$$

and by using Young's inequality

$$\frac{\partial}{\partial t}\frac{1}{2}(v - v^*)^2 \leq -\mu_2 v(v - v^*)^2(\frac{1}{2} + \chi w^* e^{-\chi(1-v^*)}) + \frac{\mu_2}{2} a_2^2 e^{2\chi v} v(w^* - w)^2.$$

After integrating over Ω we prove the lemma. □

Lemma 3.5 *Under assumption of Theorem 3.1, the following inequality holds*

$$\frac{\partial}{\partial t}\int_\Omega e^{\chi v}(w - w^*)^2 + \frac{\partial}{\partial t} A \int_\Omega (v - v^*)^2 \leq -\epsilon_3 \int_\Omega (v - v^*)^2 + e^{\chi v}(w - w^*)^2 \qquad (3.36)$$

for some positive constant A.

Proof: We multiply the both sides of inequality (3.35) by A and add it to equation (3.33) to obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega e^{\chi v}(w - w^*)^2 + \frac{\partial}{\partial t} A \int_\Omega (v - v^*)^2 + 2\int_\Omega e^{\chi v}|\nabla(w - w^*)|^2 \leq \\ -\int_\Omega e^{2\chi v}(w - w^*)^2(w(\mu_1 - \frac{\chi\mu_2}{4\underline{w}}) - A\mu_2 a_2^2 v) \\ -\int_\Omega |v - v^*|^2(Av\mu_2(1 + 2\chi w^* e^{-\chi(1-v^*)}) - \overline{h}'^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)). \end{aligned}$$

We take A such that

$$w(\mu_1 - \frac{\chi\mu_2}{4\underline{w}}) - A\mu_2 a_2^2 v > 0 \qquad (3.37)$$

and

$$(Av\mu_2(1 + 2\chi w^* e^{-\chi(1-v^*)}) - \overline{h}'^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v)) > 0. \qquad (3.38)$$

Notice that the previous inequalities are satisfied if and only if

$$\frac{w(\mu_1 - \frac{\chi\mu_2}{4\underline{w}})}{\mu_2 a_2^2 v} > A > \frac{\overline{h}'^2 e^{2\chi v} w(\mu_1 - \chi\mu_2 a_2 v))}{v\mu_2(1 + 2\chi w^* e^{-\chi(1-v^*))}}.$$

For a_2 satisfying assumption (1.22), i.e.

$$\frac{(\underline{w}\mu_1 - \frac{\chi\mu_2}{4})\underline{v}\mu_2(1+2\chi w^* e^{-\chi(1-v^*)})}{\mu_2 \overline{h}'^2 e^{2\chi\overline{w}}(\mu_1 - \chi\mu_2 a_2 \underline{v})} > a_2^2$$

we have

$$\frac{(\underline{w}\mu_1 - \frac{\chi\mu_2}{4})}{\mu_2 a_2^2} > \frac{\overline{h}'^2 e^{2\chi\overline{w}}(\mu_1 - \chi\mu_2 a_2 \underline{v})}{\underline{v}\mu_2(1+2\chi w^* e^{-\chi(1-v^*)})}$$

and there exists A satisfying (3.37) and (3.38) and $\epsilon > 0$ such that

$$\frac{d}{dt}\int_\Omega e^{\chi v}(w-w^*)^2 + \frac{\partial}{\partial t}A\int_\Omega (v-v^*)^2 \le -\epsilon\int_\Omega e^{\chi v}(w-w^*)^2 - \epsilon\int_\Omega (v-v^*)^2,$$

which ends the proof. □

Proof of Theorem 3.1
We integrate in (3.36) over $(0,t)$ to obtain, thanks to Gronwall's lemma that

$$\int_\Omega e^{\chi v}(w-w^*)^2 + \int_\Omega (v-v^*)^2 \le c_0 e^{-\epsilon t}. \tag{3.39}$$

Since

$$\begin{aligned} u - u^* &= we^{\chi v} - w^* e^{\chi v^*} \\ &= e^{\chi v}(w-w^*) + w^*(e^{\chi v} - e^{\chi v^*}) \\ &= e^{\chi v}(w-w^*) + w^* e^{\chi s}(v-v^*) \end{aligned}$$

for some $s \in (\underline{v}, \overline{v})$, it results

$$(u-u^*)^2 \le c_1 e^{\chi v}(w-w^*)^2 + c_2(v-v^*)^2 \tag{3.40}$$

for some positive constants c_1 and c_2. Thanks to (3.39) and (3.40) the proof ends. □

Remark 3.1 *By the result obtained in [7], [8] we can show global existence in time of solutions of our problem in the case $a_2 > 1$. In the same way as in section 4 of [7], [8] we have $(\log v)_t = -\mu_2 a_2 u + \mu_2(1-v)$ in (1.2), which leads to*

$$v(x,t) = v_0(x)\cdot e^{-\mu_2 a_2 \int_0^t u ds + \mu_2\int_0^t (1-v) ds}.$$

Substituting $\int_0^t u ds$ and $\int_0^t v ds$ by $k + t + \tilde{u}$ and $\tilde{v}$ respectively for any positive constant k, we have

$$v(x,t) = v_0(x)\cdot e^{-a-bt-\mu_2 a_2 \tilde{u} - \mu_2 \tilde{v}}$$

for $a = \mu_2 a_2 k > 0$ and $b = \mu_2 a_2(1 - \frac{1}{a_2}) > 0$.

Then (1.1) and (1.2) are reduced to the following single equation

$$\frac{\partial^2}{\partial t^2}\tilde{u} = \Delta\tilde{u}_t - \gamma\nabla\cdot((1+\tilde{u}_t)\nabla(v_0(x)\Theta)) + \mu_1(\tilde{u}_t + 1)(-\tilde{u}_t - a_1 v_0(x)\Theta)$$

for $\Theta = e^{-a-bt-\mu_2 a_2\tilde{u}-\mu_2\tilde{v}}$. This equation is regarded as the same type of nonlinear evolution equation (1.1) considered in [7] and [8]. Then, for sufficiently smooth initial data

$\{u_0(x), v_0(x)\}$ *assuming that* $H^m(\Omega)$*-norm of* $u_0(x) - 1$ *is sufficiently small, it is shown that for sufficiently large* a *there exist solutions*

$$u(x,t), v(x,t) \in C([0,\infty), H^{m-1}(\Omega)), \quad m \geq [n/2] + 3$$

such that $\lim_{t\to\infty} u(x,t) = 1, \lim_{t\to\infty} v(x,t) = 0$, *where* $H^m(\Omega)$ *is the Sobolev space of order* m *in* Ω.

Acknowledgments

A.Kubo is supported in part by the Grants-in-Aid for Scientific Research (C) 25400148 from Japan Society for the Promotion of Science. J.I. Tello is supported by the Project MTM2013-42907-P from MICINN (Spain).

References

[1] R. Alonso, P. Amorim and T. Goudon. *Analysis of a chemotaxis system modeling ant foraging.* Preprint 2015.

[2] H. Amann. *Highly degenerate quasilinear parabolic systems.* Ann. Scuola Norm. Sup. Pisa, Ser. IV. **18** (1991), 135–166.

[3] A. Mackey, T. Kolokolnikov and A. L. Bertozzi. *Two-species particle aggregation and stability of co-dimension one solutions*, Discrete and Continuous Dynamical Systems - Series B **19** (5) (2014), 1411–1436.

[4] C. Cosner and A.C. Lazer. *Stable Coexistence States in the Volterra–Lotka Competition Model with Diffusion.* SIAM J. Appl. Math., **44** (6) (1984), 1112–1132.

[5] A. Friedman, M.A. Fontelos and B. Hu. *Mathematical analysis of a model for the initiation of angiogenesis,* SIAM J. Math. Anal. *33* (2002), 1330–1355.

[6] A. Friedman and J.I. Tello. *Stability of solutions of chemotaxis equations in reinforced random walks.* Journal of Mathematical Analysis and Applications **272** (2002) 138–163.

[7] A. Kubo and H. Hoshino. *Nonlinear evolution equation with strong dissipation and proliferation*, "Current Trends in Analysis and its Applications". Birkhäuser, Springer (2015), 233–241.

[8] A. Kubo, H. Hoshino and K. Kimura. *Global existence and asymptotic behaviour of solutions for nonlinear evolution equations related to a tumour invasion.* Proceedings of 10th AIMS conference. To appear.

[9] A. Kubo and T. Suzuki. *Mathematical models of tumour angiogenesis.* Journal of Computational and Applied Mathematics **204** (2007) 48–55.

[10] H.A. Levine and B.D. Sleeman. *A system of reaction diffusion equation arising in the theory of reinforced random walks,* SIAM J.Appl. Math., **57** (1997), 683-730.

[11] A. Noussair, B. Ainseba and M. Bendahmane. *A reaction–diffusion system modeling predator–prey with prey-taxis.* Nonlinear Analysis: Real World Applications **9** (2008), 2086–2105.

[12] A. Muñoz and J.I. Tello. *Numerical resolution of a reinforced random walk model arising in haptotaxis.* Applied Mathematics and Computation **256** (2015), 415–424.

[13] M. Negreanu and J.I. Tello. *Asymptotic stability of a two species chemotaxis system with non-diffusive chemoattractant.* J. Differential Equations **258** (2015), 1592–1617.

[14] M. Negreanu and J.I. Tello. *On a Comparison method to reaction diffusion systems and applications.* Discrete and Continuos Dynamical Systems. Series B. **18** (2013), 2669–2688.

[15] B. D. Sleeman and H. A. Levine. *Partial differential equations of chemotaxis and angiogenesis*, Math. Methods Appl. Sci., **24** (2001), 405–426.

[16] A. Stevens. *The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems,* SIAM J. Appl. Math., **61** (2000), 183-212.

[17] T. Suzuki. *Mathematical models of tumor growth systems.* Mathematica Bohemica, **137** (2012), 201–218.

[18] T. Suzuki and R. Takahashi. *Global in time solution to a class of tumour growth systems.* Adv. Math. Sci. Appl. **19** (2009), 503–524.

[19] Y. Tao and M. Winkler. *A Chemotaxis-Haptotaxis Model: The Roles of Nonlinear Diffusion and Logistic Source.* SIAM Journal on Mathematical Analysis, **43** (2011), 685–704.

[20] J.I. Tello and M. Winkler. *Stabilization in a two-species chemotaxis system with logistic source.* Nonlinearity **25** (2012), 1413-1425.

Corresponding author:
J. Ignacio Tello
Depto. Matemática Aplicada T.I.C. E.T.S.I. Sistemas Informáticos,
Universidad Politécnica de Madrid, 28031 Madrid, Spain.
E-mail: j.tello@upm.es, Telf: +34 913365068,

Abbreviated version of the title: **A competitive-chemotactic system**