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On a Parabolic–Elliptic system with chemotaxis and logistic type growth

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Abstract

We consider a nonlinear PDEs system of two equations of Parabolic–Elliptic type with chemotactic terms. The system models the movement of a biological population "u" towards a higher concentration of a chemical agent "w" in a bounded and regular domain $\Omega \subset \mathbb{R}^N$ for arbitrary $N \in \mathbb{N}$. After normalization, the system is as follows

$$u_t - \Delta u = -div(u^m \chi \nabla w) + \mu u(1 - u^\alpha), \quad \text{in } \Omega_T = \Omega \times (0, T),$$

$$-\Delta w + w = u^\gamma, \quad \text{in } \Omega_T,$$

for some positive constants m, χ, μ, α and γ, with positive initial datum u_0 and Neumann boundary conditions.

We study the range of parameters and constrains for which the solution exists globally in time. If either

$\alpha > m + \gamma - 1$, or
$\alpha = m + \gamma - 1$ and $\mu > \frac{N\alpha - 2}{2(m-1)+N\alpha}\chi$,

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the solution exists globally in time. Moreover, if

$$\alpha \geq m+\gamma-1 \quad \text{and} \quad \mu > 2\chi,$$

and there exist positive constants $\overline{u}_0$ and $\underline{u}_0$ such that $0 < \underline{u}_0 \leq u_0 \leq \overline{u}_0 < \infty$ we have that

$$\|u-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Keywords: Parabolic–Elliptic systems; Global existence; Chemotaxis; Asymptotic behavior; Stability

1. Introduction

Chemotaxis is the ability of some living organisms to move toward a higher concentration of a chemical substance or away from it. Chemotaxis was already reported in the XIX century (see [6] and [34]), and it is present in many biological phenomena as bacteria aggregation, immune system response, angiogenesis, morphogenesis, bacterial movement, etc. (see for instance [1,4,10,13,19–21,24,25,33,41,43] among others). From the last 50 years, many of such phenomena have been described using mathematical models of PDE's, in particular, the so called Keller–Segel model (see [18]) describes the evolution of a population using a fully parabolic system of two equations

$$\begin{cases} u_t - \Delta u = -div(\chi(u)\nabla w) + g(u,w), & \text{in } \Omega_T, \\ \tau w_t - \Delta w = h(u,w), & \text{in } \Omega_T \end{cases}$$

with appropriate boundary conditions and initial data. The system has been widely studied from a mathematical point of view, see for instance [11–15,47,48] and reference therein.

Depending on the properties of the chemoattractant, the system can be simplified to a Parabolic–Elliptic system (for chemoattractant with "fast" diffusion) or to a Parabolic–ODE system (for non-diffusive chemoattractants or with slow diffusion) see for instance [7,8,22,26,28,31,32,38–40,44,45] among others. This work is focused in the case of fast chemical diffusion, where the system is simplified to the case of a Parabolic–Elliptic system, i.e. $\tau = 0$ for given functions g and h. The Parabolic–Elliptic system has been also studied from a mathematical point of view, see the works concerning blow-up of solutions [2,3,9,17,23,27,29,35–37].

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with regular boundary and g a logistic type function given by

$$g(u) := \mu u(1-u^\alpha), \quad \text{for } \alpha \geq 1$$

and

$$h(u,w) = u^\gamma - w, \quad \text{for } \gamma \geq 1.$$

The system we study in the article is the following:

$$\begin{cases} u_t - \Delta u = -div(u^m \chi \nabla w) + \mu u(1-u^\alpha), & \text{in } \Omega_T, \\ -\Delta w + w = u^\gamma, & \text{in } \Omega_T, \\ \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & \text{in } \partial\Omega \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where m, χ, μ, α and γ are positive constants.

We assume that the initial datum u_0 satisfies

$$u_0 \in W^{1,p}(\Omega) \text{ for some } p > N, \tag{1.2}$$

$$\text{there exists } \underline{u}_0 > 0 \text{ such that } u_0 \geq \underline{u}_0 \tag{1.3}$$

and the boundary condition.

$$\frac{\partial u_0}{\partial \nu} = 0 \text{ in } \partial\Omega. \tag{1.4}$$

The problem for $m = \alpha = \gamma = 1$ has been already studied in [42], where global existence and boundedness of solutions are obtained under assumptions

$$\mu > \frac{(N-2)_+}{N}\chi.$$

Moreover, for $\mu > 2\chi$ and assumptions (1.2)–(1.4), the solution satisfies

$$\|u-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} \to 0, \quad \text{as } t \to \infty. \tag{1.5}$$

The results in [42] were extended in [46] for a system with nonlinear diffusion $D(u) \leq cu^s$ for some $s \geq 0$ and $D \in C^2$ when

$$g(u) \leq a_0 - \mu u^{\alpha+1}$$

for

(i) $\alpha \geq 1$ and

$$\mu > 0 \quad \text{if } s \geq 2 - \frac{1}{N}$$

or

$$\mu > \frac{(2-s)N-2}{(2-s)N}\chi, \quad \text{if} \quad s < 2 - \frac{2}{N}.$$

(ii) $\alpha \in (0,1)$ and $s > 2 - \frac{1}{N}$.

Moreover, in [46], the authors study the asymptotic behavior of the system and (1.5) is also obtained for any $D(u) \le cu^s$ (for $s \ge 0$), $D \in C^2$ and $\mu > 2\chi$.

The case where $g = \mu u(1-u)$ and $h(u,w) = h_0(u) - w$ for a monotone increasing Lipschitz function h_0 has been recently studied in [5]. The authors prove that, under assumptions (1.2)–(1.4) and $h' < c$ the system has a uniformly bounded solution which converges in L^∞-norm to the steady state $u = 1$, $w = h(1)$.

In this article we only consider the case

$$m \ge 1 \quad \text{and} \quad \gamma \ge 1 \tag{1.6}$$

and study the global existence of solutions and its asymptotic behavior. The results are enclosed in the following theorems.

Theorem 1.1. *If either*

$$\alpha > m + \gamma - 1, \quad \textit{or} \tag{1.7}$$

$$\alpha = m + \gamma - 1 \;\; \textit{and} \;\; \mu > \frac{N\alpha - 2}{2(m-1) + N\alpha}\chi, \tag{1.8}$$

there exists a unique global in time solution to (1.1).

Theorem 1.2. *Let* $\overline{u}_0 := \max\{\sup_{x\in\bar\Omega} u_0(x), 1\}$ *and* $\underline{u}_0 := \min\{\inf_{x\in\bar\Omega} u_0(x), 1\}$*, then, if* $0 < \underline{u}_0 \le \overline{u}_0 < \infty$ *and*

$$\alpha \ge m + \gamma - 1 \quad \textit{and} \quad \mu > 2\chi,$$

the solution satisfies the asymptotic behavior

$$u \to 1, \qquad v \to 1 \qquad \textit{in } L^\infty(\Omega) \;\; \textit{as} \;\; t \to \infty. \tag{1.9}$$

2. Global existence of solutions: proof of Theorem 1.1

The proof of the theorem is divided into several steps, we first start the proof with the local existence of solutions.

Lemma 2.1. *There exists a unique solution* (u, w) *in* $(0, T_{max})$ *satisfying*

$$u, w \in C^{2+\beta, 1+\frac{\beta}{2}}_{x,t}(\Omega_{T_{max}}),$$

where T_{max} *satisfies*

$$\limsup_{t\to T_{max}} \left(\|u(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} + t\right) = \infty. \tag{2.1}$$

Moreover, the solution satisfies

$$u(t,x) \ge 0, \qquad w(t,x) \ge 0, \qquad x \in \Omega, \quad t < T_{max}. \tag{2.2}$$

Proof. The proof of local existence follows standard semigroup theory and fixed point argument; we refer to Horstmann and Winkler [16] and Tello and Winkler [42] for more details. The non-negativity of u is a consequence of the maximum principle for which also implies the non-negativity of w. $\square$

Lemma 2.2. *Suppose $\alpha > m + \gamma - 1$. Then for any $\delta > 1$ there exists $c = c(\delta, \|u_0\|_{L^\delta(\Omega)}) > 0$ such that*

$$\|u(t)\|_{L^\delta(\Omega)} \le c, \quad \forall t \in (0, T_{max}) \tag{2.3}$$

as well as

$$\int_0^T \int_\Omega u^{\alpha+\delta} + \int_0^T \int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le c(T+1), \quad \forall T \in (0, T_{max}) \tag{2.4}$$

hold.

Proof. We multiply the first equation in (1.1) by $u^{\delta-1}$ and integrate by parts over Ω to obtain

$$\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + (\delta-1)\int_\Omega u^{\delta-2}|\nabla u|^2 = (\delta-1)\chi\int_\Omega u^{m+\delta-2}\nabla u\cdot\nabla w + \mu\int_\Omega u^\delta(1-u^\alpha). \tag{2.5}$$

In the same way, we multiply the second equation in (1.1) by $u^{m+\delta-1}$, to obtain, after integration by parts

$$(m+\delta-1)\int_\Omega u^{m+\delta-2}\nabla u\cdot\nabla w = -\int_\Omega u^{m+\delta-1}w + \int_\Omega u^{m+\delta+\gamma-1} \tag{2.6}$$

for $t \in (0, T_{max})$. Inserting (2.6) into (2.5) it yields

$$\begin{aligned} &\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + (\delta-1)\int_\Omega u^{\delta-1}|\nabla u|^2 = \\ &= \frac{(\delta-1)\chi}{m+\delta-1}\left(-\int_\Omega u^{m+\delta-1}w + \int_\Omega u^{m+\delta+\gamma-1}\right) + \mu\int_\Omega u^\delta(1-u^\alpha) \\ &\le \frac{(\delta-1)\chi}{m+\delta-1}\int_\Omega u^{m+\delta+\gamma-1} + \mu\int_\Omega u^\delta - \mu\int_\Omega u^{\delta+\alpha}. \end{aligned} \tag{2.7}$$

We now consider the term $\int_\Omega u^{m+\delta+\gamma-1}$, since $\alpha > m+\gamma-1$ we have that $\alpha+\delta > m+\delta+\gamma-1$ and thanks to Young's inequality

$$\int_\Omega u^{m+\delta+\gamma-1} \le \epsilon_1\int_\Omega u^{\alpha+\delta} + c_1(\epsilon_1, |\Omega|, \delta, \alpha, \gamma, m)$$

and

$$\int_\Omega u^\delta \le \epsilon_2 \int_\Omega u^{\alpha+\delta} + c_2(\epsilon_2, |\Omega|, \delta, \alpha, \gamma, m)$$

for arbitrary positive constants ϵ_1 and ϵ_2. Then, for $\epsilon < \mu$ we have that

$$-(\mu - \epsilon)\int_\Omega u^{\delta+\alpha} \le -\frac{(\delta-1)\chi}{m+\delta-1}\int_\Omega u^{m+\delta+\gamma-1} - \mu \int_\Omega u^\delta + c(\epsilon, |\Omega|, \delta, \alpha, \gamma, m)$$

which proves

$$\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + \frac{4(\delta-1)}{\delta^2}\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le -\epsilon \int_\Omega u^{\alpha+\delta} + c(\epsilon, |\Omega|, \delta, \alpha, \gamma, m). \tag{2.8}$$

We apply the Hölder inequality to the term $\int_\Omega u^{\alpha+\delta}$ to obtain

$$\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + \frac{4(\delta-1)}{\delta^2}\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le -\frac{\epsilon}{|\Omega|^{\frac{\alpha}{\delta}}}\left(\int_\Omega u^\delta\right)^{\frac{\alpha+\delta}{\delta}} + c(\epsilon). \tag{2.9}$$

We denote $y(t) := \int_\Omega u^\delta(t)$ which satisfies

$$y' \le -c_3 y^{\frac{\alpha+\delta}{\delta}} + c_4 \text{ in } (0, T_{max})$$

with positive constants c_3 and c_4 depending on α, δ, ϵ and $|\Omega|$. Then

$$y(t) \le \max\left\{y_0, \left(\frac{c_4}{c_3}\right)^{\frac{\delta}{\alpha+\delta}}\right\} \text{ for all } t \in [0, T_{max})$$

which yields (2.3). We integrate over $(0, T_{max})$ in (2.8) to conclude (2.4). □

Lemma 2.3. *If $\alpha = m + \gamma - 1$, then, for any $\delta \in \left(1, \frac{(m-1)\mu+\chi}{(\chi-\mu)_+}\right)$ we have that*

$$\|u(t)\|_{L^\delta(\Omega)} \le c, \quad \forall t \in (0, T_{max}) \tag{2.10}$$

as well as

$$\int_0^T\int_\Omega u^{\alpha+\delta} + \int_0^T\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le c(T+1), \quad \forall T \in (0, T_{max}) \tag{2.11}$$

hold.

Moreover, if in addition

$$\mu > \frac{N\alpha - 2}{2(m-1) + N\alpha}\chi \tag{2.12}$$

then (2.10) holds for any $\delta > 1$.

Proof. As in the previous lemma, we multiply the first equation in (1.1) by $u^{\delta-1}$ and integrate by parts over Ω to obtain

$$\begin{aligned} &\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + (\delta-1)\int_\Omega u^{\delta-2}|\nabla u|^2 = \\ &= \frac{(\delta-1)\chi}{m+\delta-1}\left(-\int_\Omega u^{m+\delta-1}w + \int_\Omega u^{m+\delta+\gamma-1}\right) + \mu\int_\Omega u^\delta(1-u^\alpha) \\ &\le \frac{(\delta-1)\chi}{m+\delta-1}\int_\Omega u^{m+\delta+\gamma-1} + \mu\int_\Omega u^\delta - \mu\int_\Omega u^{\delta+\alpha}. \end{aligned} \tag{2.13}$$

We set $\epsilon := \frac{1}{2}\left(\mu - \frac{(\delta-1)\chi}{m+\delta-1}\right)$, which is positive for $\delta \in \left(1, 1 + \frac{m\mu}{(\chi-\mu)_+}\right)$. Thanks to Young's inequality we estimate

$$\mu\int_\Omega u^\delta \le \epsilon\int_\Omega u^{\alpha+\delta} + c(\epsilon)$$

to obtain

$$\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + \frac{4(\delta-1)}{\delta^2}\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le -\epsilon\int_\Omega u^{\alpha+\delta} + c(\epsilon). \tag{2.14}$$

We apply the Hölder inequality to the term $\int_\Omega u^{\alpha+\delta}$ to obtain

$$\frac{1}{\delta}\frac{d}{dt}\int_\Omega u^\delta + \frac{4(\delta-1)}{\delta^2}\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le -\frac{\epsilon}{|\Omega|^{\frac{\alpha}{\delta}}}\left(\int_\Omega u^\delta\right)^{\frac{\alpha+\delta}{\delta}} + c(\epsilon), \tag{2.15}$$

and proceed as in the previous lemma to conclude (2.11).

To prove (2.11) when μ satisfies (2.12) we proceed as in Lemma 2.3 in Tello and Winkler [42]. Since the case $\chi \le \mu$ is included in the previous one, we focus on the case

$$\chi > \mu.$$

We introduce the auxiliary unknown $v = u^{\frac{\delta}{2}}$ which satisfies the inequality

$$\frac{1}{\delta}\frac{d}{dt}\int_\Omega v^2 + \frac{4(\delta-1)}{\delta^2}\int_\Omega |\nabla v|^2 \le \left(\frac{(\delta-1)\chi}{m+\delta-1} - \mu\right)\int_\Omega v^{\frac{2\alpha}{\delta}+2} + \mu\int_\Omega v^2. \tag{2.16}$$

By the Gagliardo–Nirenberg Inequality we know that

$$\|v\|_{L^p(\Omega)} \le C\|v\|^{1-a}_{L^q(\Omega)}\|v\|^a_{W^{1,2}(\Omega)}$$

for

$$\frac{1}{p} = \left(\frac{1}{2} - \frac{1}{N}\right)a + \frac{1-a}{q}, \quad \text{i.e.} \quad a = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} - \frac{1}{2} + \frac{1}{N}}.$$

We take

$$p = 2(1+\alpha/\delta), \quad q = 2 \quad \text{and} \quad a = \frac{\frac{1}{2} - \frac{1}{2+\frac{2\alpha}{\delta}}}{\frac{1}{2} - \frac{1}{2} + \frac{1}{N}} = \frac{N}{2} - \frac{N}{2+\frac{2\alpha}{\delta}} = \frac{\frac{N\alpha}{\delta}}{2+\frac{2\alpha}{\delta}} = \frac{N\alpha}{2(\delta+\alpha)}.$$

Notice that assumption (2.12) is equivalent to

$$(N\alpha - 2)\chi < \mu(2(m-1) + N\alpha)$$

i.e.

$$N\alpha(\chi - \mu) < 2(\mu(m-1) + \chi)$$

or

$$\frac{N\alpha(\chi-\mu)}{\mu(m-1)+\chi} < 2. \tag{2.17}$$

Since

$$a = \frac{N\alpha}{2(\delta+\alpha)}$$

we have that

$$\frac{a(\chi-\mu)2(\delta+\alpha)}{(m-1)\mu+\chi} < 2$$

i.e.

$$a < \frac{(m-1)\mu+\chi}{(\chi-\mu)(\delta+\alpha)}$$

taking $\delta > \max\{\frac{(m-1)\mu+\chi}{\chi-\mu} - \alpha, \frac{N\alpha}{2}\}$ we have that

$$a < 1$$

and

$$a\left(\frac{2\alpha}{\delta}+2\right) = \frac{2a}{\delta}(\alpha+\delta) = \frac{N\alpha}{\delta} < 2.$$

Then

$$\int_\Omega v^{\frac{2\alpha}{\delta}+2} = \|v\|^{\frac{2\alpha}{\delta}+2}_{L^{\frac{2\alpha}{\delta}+2}(\Omega)} \le c(\|v\|_{L^2(\Omega)})\|v\|^{a\left(\frac{2\alpha}{\delta}+2\right)}_{H^1(\Omega)} \le c_3\left[\|\nabla v\|_{L^2(\Omega)}+1\right]^{a\left(\frac{2\alpha}{\delta}+2\right)}$$

and thanks to (2.17) we have, for some $c_4 > 0$ and arbitrary $\epsilon > 0$,

$$\int_\Omega v^{\frac{2\alpha}{\delta}+2} \le c_3\|\nabla v\|^{2-\epsilon_3}_{L^2(\Omega)} + c_4 \le \epsilon\|\nabla v\|^2_{L^2(\Omega)} + c_5(\epsilon).$$

We substitute the previous inequality into (2.16) to get

$$\frac{d}{dt}\int_\Omega u^\delta + c_6\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le c_7 + \mu\int_\Omega u^\delta(1-u).$$

Thanks to Young's Inequality we get

$$\frac{d}{dt}\int_\Omega u^\delta + c_6\int_\Omega |\nabla u^{\frac{\delta}{2}}|^2 \le \mu\left(c_8 - \int_\Omega u^\delta\right). \tag{2.18}$$

Maximum principle for ODE gives that

$$\int_\Omega u^\delta \le \max\{\|u_0\|^\delta_{L^\delta(\Omega)}, c_8\}$$

which ends the proof. □

End of the proof of the Theorem 1.1. To end the proof of the Theorem 1.1 we notice that u satisfies

$$u_t - \Delta u = -m\chi u^{m-1}\nabla u\nabla w + \chi u^m(u^\gamma - w) + \mu u(1-u^\alpha).$$

Since u is uniformly bounded in $L^p(\Omega)$ for any $p < \infty$ and α, m and γ satisfying the assumptions of Theorem 1.1, we have that

$$\|w\|_{W2,p(\Omega)} \le c, \qquad \text{for some} \qquad p > N$$

and

$$\|w\|_{W^{1,\infty}(\Omega)} \le c \qquad \text{uniformly}.$$

Then

$$-m\chi u^{m-1}\nabla u \nabla w + \chi u^m (u^\gamma - w) + \mu u(1-u^\alpha) \in L^s(\Omega_T)$$

for any $s < 2$ and $T < T_{max}$. Standard regularity theory gives the global existence of solutions. □

3. Asymptotic behavior of the solutions

In this section we prove Theorem 1.2 by using a rectangle method, i.e. we construct a sub- and a super-solution which are homogeneous in space. Such functions are obtained as solutions of the following ODE's system

$$\begin{cases} \overline{u}_t = \chi \overline{u}^m(\overline{u}^\gamma - \underline{u}^\gamma) + \mu\overline{u}(1-\overline{u}^\alpha), & \overline{u}(0) > \overline{u}_0 := \max\{\max_{x\in\bar\Omega} u_0(x), 1\}, \\ \underline{u}_t = \chi \underline{u}^m(\underline{u}^\gamma - \overline{u}^\gamma) + \mu\underline{u}(1-\underline{u}^\alpha), & \underline{u}(0) < \underline{u}_0 := \inf\{\min_{x\in\bar\Omega} u_0(x), 1\}. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $\underline{u}_0$ and $\overline{u}_0$ be positive initial data, satisfying*

$$0 < \underline{u}_0 < 1 < \overline{u}_0, \tag{3.2}$$

then, there exists a unique solution to (3.1) in $(0, T'_{max})$ satisfying

$$0 < \underline{u} < 1 < \overline{u}, \qquad \text{for } t \in (0, T'_{max}), \tag{3.3}$$

where T'_{max} is defined by

$$\limsup_{t\to T'_{max}} \big(|\overline{u}| + |\underline{u}| + t\big) = \infty. \tag{3.4}$$

Proof. We split the proof into several steps:

- **Step 1. Local existence and uniqueness of solutions.**
  Since the right-hand side terms of (3.1) are continuous and locally Lipschitz functions of $\underline{u}$ and $\overline{u}$ we have that there exists a unique solution in the interval $[0, T'_{max})$ for T'_{max} defined in (3.4). We also notice that, since the right-hand side terms are $C^1(\mathbb{R}^2)$ the solutions $(\underline{u}, \overline{u}) \in [C^2(0, T'_{max}]^2$.
- **Step 2. The solution satisfies (3.3).**
  By contradiction, we assume that there exists a positive $t_0 < T'_{max}$ such that (3.3) is satisfied for any $t < t_0$ and at $t = t_0$ (3.3) fails, then:
  – If $\underline{u}(t_0) = 0$, by uniqueness of solution, the backward solution $\underline{u}(t) = 0$ for any $t \in [0, t_0]$ which contradicts (3.2) and proves

$$\underline{u}(t_0) > 0. \tag{3.5}$$

– If $\underline{u}(t_0) = 1$ and $\overline{u}(t_0) > 1$, we have that

$$\underline{u}'(t_0) = \chi \underline{u}^m(t_0)(\underline{u}^\gamma(t_0) - \overline{u}^\gamma(t_0)) + \mu \underline{u}(t_0)(1 - \underline{u}^\alpha(t_0)) < 0$$

which contradicts $\underline{u}(t_0) = 1$ and proves

$$\underline{u}(t_0) < 1 \quad \text{if } \overline{u}(t_0) > 1. \tag{3.6}$$

– The case $\overline{u}(t_0) = 1$ and $\underline{u}(t_0) < 1$ is similar to the previous one and therefore

$$\overline{u}(t_0) > 1 \quad \text{if } \underline{u}(t_0) < 1. \tag{3.7}$$

– If

$$\underline{u}(t_0) = 1 = \overline{u}(t_0) \tag{3.8}$$

the backward solution of the system satisfies $\underline{u}(t) = 1 = \overline{u}(t)$ for $t \in [0, t_0)$, uniqueness of solution contradicts (3.2) and the proof ends. □

Lemma 3.2. *Under assumption*

$$\alpha \geq m + \gamma - 1 \quad \text{and} \quad \mu > 2\chi, \tag{3.9}$$

we have that $T_{max} = \infty$ and the solution to (3.1) satisfies

$$\overline{u} \to 1, \qquad \underline{u} \to 1 \qquad \text{for} \quad t > 0. \tag{3.10}$$

Proof. We divide the first equation by $\overline{u}$ and the second by $\underline{u}$ to obtain

$$\frac{\overline{u}_t}{\overline{u}} = \chi \overline{u}^{m-1}(\overline{u}^\gamma - \underline{u}^\gamma) + \mu(1 - \overline{u}^\alpha),$$
$$\frac{\underline{u}_t}{\underline{u}} = \chi \underline{u}^{m-1}(\underline{u}^\gamma - \overline{u}^\gamma) + \mu(1 - \underline{u}^\alpha).$$

We subtract both equations to get:

$$\frac{d}{dt}\left(\ln \overline{u} - \ln \underline{u}\right) = \chi\left(\overline{u}^{m-1}(\overline{u}^\gamma - \underline{u}^\gamma) - \underline{u}^{m-1}(\underline{u}^\gamma - \overline{u}^\gamma)\right) + \mu(\underline{u}^\alpha - \overline{u}^\alpha),$$
$$\frac{d}{dt}\left(\ln \overline{u} - \ln \underline{u}\right) = \chi(\overline{u}^{m-1+\gamma} - \underline{u}^{m-1+\gamma}) - \chi(\overline{u}^{m-1}\underline{u}^\gamma - \underline{u}^{m-1}\overline{u}^\gamma) + \mu(\underline{u}^\alpha - \overline{u}^\alpha), \tag{3.11}$$

and since $\underline{u} < 1 < \overline{u}$ we have

$$-\chi(\overline{u}^{m-1}\underline{u}^\gamma - \underline{u}^{m-1}\overline{u}^\gamma) \leq \chi(\overline{u}^{m+\gamma-1} - \underline{u}^{m+\gamma-1}).$$

Since

$$\alpha \geq m + \gamma - 1$$

and $\overline{u} \geq 1$ we have that

$$\overline{u}^{m+\gamma-1} \leq \overline{u}^{\alpha},$$

in the same way, since $\underline{u} \leq 1$ we also get

$$\underline{u}^{m+\gamma-1} \geq \underline{u}^{\alpha},$$

then

$$\chi(\overline{u}^{m+\gamma-1} - \underline{u}^{m+\gamma-1}) \leq \chi(\overline{u}^{\alpha} - \underline{u}^{\alpha})$$

and thanks to (3.11) we deduce

$$\frac{d}{dt}\left(\ln \overline{u} - \ln \underline{u}\right) \leq (2\chi - \mu)(\overline{u}^{\alpha} - \underline{u}^{\alpha}).$$

Notice that, under assumptions (3.9) and (3.3) we have

$$\frac{d}{dt}\left(\ln \overline{u} - \ln \underline{u}\right) \leq 0.$$

After integration it results

$$\ln \overline{u} - \ln \underline{u} \leq \ln \overline{u}_0 - \ln \underline{u}_0$$

and since $\overline{u} \geq 1$ (see Lemma 3.1) we obtain a lower bound for $\underline{u}$

$$\underline{u} \geq \frac{\underline{u}_0}{\overline{u}_0}.$$

Since

$$(2\chi - \mu)(\overline{u}^{\alpha} - \underline{u}^{\alpha}) = (2\chi - \mu)\alpha\xi^{\alpha-1}(\overline{u} - \underline{u})$$

for some $\xi \geq \frac{\underline{u}_0}{\overline{u}_0}$ we have that

$$(2\chi - \mu)\alpha\xi^{\alpha-1} < -\epsilon_0, \qquad \text{for some} \qquad \epsilon_0 > 0. \tag{3.12}$$

So, we have

$$\frac{d}{dt}\left(\ln \overline{u} - \ln \underline{u}\right) \leq -\epsilon_0(\overline{u} - \underline{u}) \leq -\epsilon_0\xi_2\left(\ln \overline{u} - \ln \underline{u}\right),$$

for some $\xi_2 \geq \frac{\underline{u}_0}{\overline{u}_0}$. By integration, we prove that $\overline{u}$ is uniformly bounded by a constant which depends on the initial data and therefore $T'_{\max} = \infty$. Thanks to Gronwall's Lemma and Lemma 3.1 we end the proof. □

Lemma 3.3. *Under assumption (1.2), (1.3) the solution to (1.1) satisfies:*

$$\underline{u}(t) < u(x,t) < \overline{u} \qquad \textit{for} \quad t < T_{max}.$$

Proof. The first equation in (1.1) can be expressed as

$$u_t - \Delta u = -m\chi u^{m-1}\nabla u \nabla w + \chi u^m(u^\gamma - w) + \mu u(1-u^\alpha). \tag{3.13}$$

We now proceed as in Tello and Winkler [42] (see also [30]) and construct the auxiliary functions

$$\overline{U} = u - \overline{u}, \quad \underline{U} = u - \underline{u}, \quad \overline{W} = w - \overline{u}^\gamma, \quad \underline{W} = w - \underline{u}^\gamma.$$

We define $g(u) = \chi u^{m+\gamma} + \mu u(1-u^\alpha)$, then $\overline{U}$ satisfies

$$\overline{U}_t - \Delta\overline{U} = -m\chi u^{m-1}\nabla\overline{U}\cdot\nabla w + g(u) - g(\overline{u}) + \chi(\overline{u}^m\underline{u}^\gamma - u^m w).$$

We multiply the previous equation by the positive part of $\overline{U}$ and integrate over Ω to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \overline{U}_+^2 + \int_\Omega |\nabla\overline{U}_+|^2 = -\int_\Omega m\chi u^{m-1}\nabla\overline{U}\cdot\nabla w\,\overline{U}_+ + \int_\Omega (g(u)-g(\overline{u}))\overline{U}_+ + \int_\Omega \chi(\overline{u}^m\underline{u}^\gamma - u^m w)\overline{U}_+.$$

We consider the right-hand side terms separately:

$$-\int_\Omega m\chi u^{m-1}\nabla\overline{U}\cdot\nabla w\,\overline{U}_+, \quad \int_\Omega (g(u)-g(\overline{u}))\overline{U}_+, \quad \int_\Omega \chi(\overline{u}^m\underline{u}^\gamma - u^m w)\overline{U}_+.$$

So, for any $t \le T < T_{max}$, we have that u is uniformly bounded in $L^\infty(\Omega_T)$ and $w(t) \in W^{1,\infty}(\Omega)$. Then, for $t \le T$ we have that

(i)

$$-\int_\Omega m\chi u^{m-1}\nabla\overline{U}\cdot\nabla w\,\overline{U}_+ \le c_0\|u\|^{2m-2}_{L^\infty(\Omega_T)}\|w\|^2_{W^{1,\infty}(\Omega)}\int_\Omega \overline{U}_+^2 + \frac{1}{2}\int_\Omega |\nabla\overline{U}_+|^2.$$

(ii)

$$\int_\Omega (g(u)-g(\overline{u}))\overline{U}_+ = \int_\Omega g'(\xi)\overline{U}_+^2 \le c(\|u\|_{L^\infty(\Omega_T)}, g')\int_\Omega \overline{U}_+^2.$$

(iii)

$$\int_\Omega \chi(\overline{u}^m \underline{u}^\gamma - u^m w)\overline{U}_+ = \int_\Omega \chi((\overline{u}^m - u^m)\underline{u}^\gamma - u^m(w - \underline{u}^\gamma))\overline{U}_+$$
$$= \int_\Omega \chi((\overline{u}^m - u^m)\underline{u}^\gamma - u^m(w - \underline{u}^\gamma))\overline{U}_+$$

since $(\overline{u}^m - u^m)\overline{U}_+ \leq 0$ and $w - \underline{u}^\gamma = \underline{W}$ we have

$$\int_\Omega \chi(\overline{u}^m \underline{u}^\gamma - u^m w)\overline{U}_+ \leq -\chi \int_\Omega u^m \underline{W}\overline{U}_+$$
$$\leq \chi \|u\|^m_{L^\infty(\Omega)} \int_\Omega \underline{W}_- \overline{U}_+$$
$$\leq \frac{1}{2}\chi \|u\|^m_{L^\infty(\Omega)} \int_\Omega \underline{W}^2_- + \overline{U}^2_+$$

where $(\,\cdot\,)_-$ is the negative part function defined as $(s)_- = (-s)_+$.
Multiplying the equation

$$-\Delta \underline{W} + \underline{W} = u^\gamma - \underline{u}^\gamma$$

by $-\underline{W}_-$ and integrating over Ω, by Young's inequality, it results

$$\int_\Omega |\nabla \underline{W}_-|^2 + \frac{1}{2}\int_\Omega |\underline{W}_-|^2 \leq \frac{1}{2}\int_\Omega (u^\gamma - \underline{u}^\gamma)^2_-.$$

Thanks to assumption (1.6) we get

$$\int_\Omega (u^\gamma - \underline{u}^\gamma)^2_- \leq c(\gamma, \|u\|_{L^\infty(\Omega)}) \int_\Omega \underline{U}^2_-.$$

So we have

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \overline{U}^2_+ + \frac{1}{2}\int_\Omega |\nabla \overline{U}_+|^2 \leq c(\|u\|_{L^\infty(\Omega)}, g') \int_\Omega \overline{U}^2_+ + \underline{U}^2_-. \tag{3.14}$$

In the same way we obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \underline{U}^2_- + \frac{1}{2}\int_\Omega |\nabla \underline{U}_-|^2 \leq c(\|u\|_{L^\infty(\Omega)}, g') \int_\Omega \overline{U}^2_+ + \underline{U}^2_-. \tag{3.15}$$

Adding (3.14) and (3.15) and thanks to Gronwall's Lemma we obtain

$$\overline{U}_+ = \underline{U}_- = 0, \qquad \text{for any } t \le T < T_{\max}.$$

The proof ends taking limits when $T \to T_{max}$. □

End of the proof of the Theorem 1.2. We have seen in Lemma 3.3 that $\underline{u} \le u \le \overline{u}$ and $\underline{u}^m \le w \le \overline{u}^m$, since $T'_{\max} = \infty$ we obtain that u and w are uniformly bounded in $(0, \infty)$. Notice that

$$\|u - 1\|_{L^\infty(\Omega)} \le |\overline{u} - 1| + \|\overline{u} - u\|_{L^\infty(\Omega)} \le |\overline{u} - \underline{u}|,$$
$$\|w - 1\|_{L^\infty(\Omega)} \le |\overline{u}^m - 1| + \|\overline{u}^m - w\|_{L^\infty(\Omega)} \le |\overline{u}^m - \underline{u}^m|.$$

Taking limits in the previous inequalities we end the proof of Theorem 1.2. □

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