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AN INVERSE PROBLEM FOR THE COMPRESSIBLE REYNOLDS EQUATION

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Abstract. We study the existence of solutions to a system of equations for equilibrium positions in lubricated journal bearings under load effects. The mechanism under consideration consists of two parallel cylinders, one inside the other, in close distance and relative motion. The unknowns of the problem are the equilibrium position of the inner cylinder and the pressure of the lubricant described by the compressible Reynolds equation. To complete the system, Newton's second law gives the equilibrium of forces. We present results on existence of solutions for a range of applied forces F.

1. Introduction. We consider a lubricated journal bearing system for compressible fluids. The system consists of two parallel cylinders, one inside the other in close proximity, where a shaft or journal (the inner cylinder) rotates freely in a shell (the exterior cylinder). A compressible fluid, the lubricant, fills the gap between the cylinders to avoid contact. An external force $F = (F_1, F_2) \in \mathbb{R}^2$ is applied on the inner cylinder, which turns with a known velocity, assumed constant.

The external force produces a displacement of the shaft that generates a high pressure region. That high pressure originates a hydrodynamic force that counteracts the external force. We assume that the distance "h" between the two cylinders satisfies the thin-film hypothesis, so that the pressure does not depend on the normal coordinate to the

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cylinders and satisfies the compressible Reynolds equation

$$\begin{cases} -\mathrm{div}(Q(hp)h^3 p\nabla p) = -6\nu U \dfrac{\partial}{\partial\theta}(hp) & \text{in } (0,2\pi]\times(0,1), \\ p = p_a & \text{on } x=0 \quad \text{and} \quad x=1, \end{cases}$$

where Q is the relative flow rate, ν is the lubricant viscosity (assumed constant), U is the velocity of the surface and p_a is the atmospheric pressure. We assume that ν, U and p_a are positive given constants.

As a first approximation, Q is considered constant in the "continuum Reynolds equation" (see [2]). Subsequent models use better approximations of Q, assuming it to be dependent on the magnitude of "h". In this paper we consider one such model, the so-called *first-order slip model* (see Burgdorfer [3]):

$$Q(ph) = 1 + \frac{\tilde{\lambda}}{ph}, \tag{1.1}$$

where $\tilde{\lambda}$ is the molecular mean free path of air. This model allows us to take into account the rarified gas effects when the distance h is smaller than $0.1\mu m$.

Some other alternative expressions for Q have been proposed in the literature. For instance, the second-order slip model, where Q presents a second order term, i.e.

$$Q(ph) = 1 + \frac{\tilde{\lambda}}{ph} + \frac{a_1}{(ph)^2}$$

or the Fukui Kaneko model (see [8]).

In this problem, the position of the shaft is unknown and is defined by the displacement of its center. Since we only consider displacements which keep the cylinders on parallel, the position is defined by two parameters, the radial coordinate of the displacement "η" ((η_1,η_2) in cartesian coordinates) and the angular coordinate "α". The coefficient "h" approximates the distance between the cylinders, and after renormalization it is given by

$$h(\theta,\eta,\alpha) = 1 - \eta\cos(\theta-\alpha) = 1 - \eta_1\cos\theta - \eta_2\sin\theta. \tag{1.2}$$

The problem we address in this paper is to find the shaft equilibrium position "(η,α)" and the pressure "p" of the lubricant for a given constant force $F=(F_1,F_2)\in\mathbb{R}^2$, such that

$$p : (\theta, x) \in \Omega \to \mathbb{R}$$

satisfies the compressible Reynolds equation, which, after renormalization, is given by

$$\begin{cases} -div((h^3p+\lambda h^2)\nabla p) = -\Lambda\dfrac{\partial hp}{\partial\theta}, & (\theta,x)\in\Omega, \\ p \text{ is } 2\pi-\text{periodic in } \theta, & \\ p(\theta,0)=p(\theta,1)=p_a, & \theta\in[0,2\pi), \end{cases} \tag{1.3}$$

where h is defined in (1.2), $\Omega=(0,2\pi]\times(0,1)\subset\mathbb{R}^2$, $\lambda=\tilde{\lambda}/h_0$, Λ is a nondimensional constant defined by

$$\Lambda := \frac{6\nu U}{(R_1-R_0)^2}$$

and R_1-R_0 is the difference between the radii of the cylinders.

To set the problem we have to take into account the balance of forces acting on the journal. If we denote by $s = (s_1(t), s_2(t))$ the displacement in cartesian coordinates, thanks to Newton's second law we have

$$m\frac{ds_1}{dt} = F_1 - \int_\Omega p\cos\theta d\theta dx, \qquad m\frac{ds_2}{dt} = F_2 - \int_\Omega p\sin\theta d\theta dx.$$

Since at the equilibrium position there is no displacement, i.e. $s' = 0$, we have

$$\int_\Omega p\cos\theta d\theta dx = F_1, \qquad \int_\Omega p\sin\theta d\theta dx = F_2. \tag{1.4}$$

To the best of the authors' knowledge, only a few works studying the existence of inverse problems in lubricated devices for compressible fluids have been published. In the literature we can find some results for sliders, i.e. systems where a surface slides above a plane; see Hafidi [9] for more details. For the incompressible case there is a larger number of references, mainly concerning numerical simulations. See for instance [7], [5], [6] and the survey [1] for more details.

Below, we present the main result of the article.

THEOREM 1.1. There exists $\overline{\epsilon} > 0$ such that, for any $F \in \mathbb{R}^2$ satisfying

$$|F| = |F_1^2 + F_2^2|^{\frac{1}{2}} < \overline{\epsilon}, \tag{1.5}$$

the system (1.3), (1.4) has at least one weak solution.

This result means that the set of admissible external forces (those for which the existence of solutions takes place) contains the range of forces whose magnitude does not exceed a certain threshold $\overline{\epsilon}$. The definition of $\overline{\epsilon}$ is given in the proof of the theorem (see (2.5)) and it is defined in terms of integrals. Clearly, problem (1.3), (1.4) admits a solution for any $F \in \mathbb{R}^2$ if $\overline{\epsilon} = \infty$. The question of the boundedness of $\overline{\epsilon}$ remains open and the answer may depend on the geometry of h.

In the incompressible case (see [5]) it is possible to compute explicitly $\overline{\epsilon}$ by using a sub- and super-solution method for a large group of geometries. In our problem that method cannot be applied directly due to the nonlinear terms of the compressible Reynolds equation.

The notion of solution used in Theorem 1.1 corresponds to the standard definition of weak solution for the Reynolds equation. For the reader's convenience, in the following section we include that definition and also some known results on degree theory that are used in Section 3 to prove Theorem 1.1.

2. Preliminaries. We first start with the notion of admissible displacements of the inner cylinder, defined through the following set:

$$A := \Big\{(\eta_1, \eta_2) \in \mathbb{R}^2 : \quad \eta_1^2 + \eta_2^2 < 1\Big\} = \Big\{(\eta, \alpha) \in \mathbb{R}_+ \times [0, 2\pi), \quad \eta < 1\Big\}.$$

Notice that the function h defined in (1.2) is positive for every $\theta \in [0, 2\pi)$ if and only if $(\eta_1, \eta_2) \in A$.

Let us define the following functional space:

$$V(\Omega) := \{\phi \in H^1(\Omega) : \quad \phi \text{ is } 2\pi\text{-periodic in } \theta \text{ and } \phi(\theta, 0) = \phi(\theta, 1) = 0\}$$

with the norm

$$\|\phi\|_V := \int_\Omega |\nabla\phi|^2.$$

For the known standard notion of weak solutions we have

DEFINITION 2.1. A weak solution to (1.3), (1.4) is a triplet (p, η, α), such that

$$p - p_a \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad p \geq 0,$$

$(\eta, \alpha) \in A$ and

$$\int_\Omega (h^3(\eta,\alpha)p + \lambda h^2(\eta,\alpha))\nabla p \nabla\phi = \Lambda \int_\Omega ph(\eta,\alpha)\frac{\partial\phi}{\partial\theta}, \tag{2.1}$$

for any $\phi \in V(\Omega)$, where p also satisfies (1.4) for $h(\eta, \alpha)$ given in (1.2).

Some known results in degree theory.

For completeness, we briefly cite some definitions and results that are used in the last section to prove Theorem 1.1.

Let $S \subset \mathbb{R}^n$ be a bounded open subset with regular boundary ∂S and let $f : S \to \mathbb{R}^n$ be a $C^1(S)$ function. Let $y_0 \in \mathbb{R}^n$ be such that $y_0 \notin f(\partial S)$ and $Df(x)$ is invertible for all $x \in f^{-1}(y_0)$. Then, $f(x) = y_0$ has either no solutions in S or a finite number of solutions: $x_1, x_2, \cdots, x_r$ with $det(Df(x_i)) \neq 0$ for $i = 1, 2, \cdots, r$.

DEFINITION 2.2. The degree of the function f in the subset S at the point y_0 is defined as

$$deg(f, S, y_0) := \begin{cases} 0, & \text{if } r = 0, \\ \displaystyle\sum_{i=1}^r sign(det(Df(x_i))), & \text{if } r \neq 0. \end{cases}$$

REMARK 2.3. In the particular case of a linear function $f(x) = Ax$ with $det(A) \neq 0$, the previous definition may be written as

$$deg(Ax, S, y_0) := \begin{cases} 0, & \text{if } y_0 \notin f(S), \\ sign(\det A), & \text{if } y_0 \in f(S). \end{cases}$$

The previous definition is extended to continuous maps; see for instance [10, Extension Lemma, p. 60]. The following result is used in the proof of the theorem (see [10, Corollary 4 to Theorem 1.12, p. 81] for more details).

THEOREM 2.4. Let S be a bounded open set in $\mathbb{R}^n$. Let f_0 and f_1 be continuous functions

$$f_i : \overline{S} \to \mathbb{R}^n \quad \text{for } i = 0, 1.$$

We also assume that S is a star-shaped domain with respect to $y_0 \in S$ such that

$$[f_0(x) - y_0] \cdot [f_1(x) - y_0]^t > 0, \text{ for any } x \in \partial S,$$

where v^t denotes the transpose vector of v. Then,

$$deg(f_0, S, y_0) = deg(f_1, S, y_0).$$

REMARK 2.5. Notice that if $deg(f, S, y_0) \neq 0$, we obtain the existence of at least one solution $x \in S$ to the equation $f(x) = y_0$.

Continuous dependence of p with respect to h.

In the following we prove a lemma concerning the continuity of a solution of (1.3) (i.e. the direct problem) with respect to h.

LEMMA 2.6. Let $h_i \in L^\infty(\Omega)$, for $i \in \mathbb{N}$, such that

$$0 < \underline{h} \leq h_i(x) \leq \overline{h} < \infty, \quad x \in \Omega, \tag{2.2}$$

for some positive constants $\underline{h}$ and $\overline{h}$ such that

$$h_i \to h_0, \quad \text{in } \ L^\infty(\Omega), \quad \text{as } \ i \to \infty.$$

Let p_i be the solution of

$$\begin{cases} -div((h_i^3 p_i + \lambda h_i^2)\nabla p_i) = -\Lambda \dfrac{\partial h_i p_i}{\partial \theta}, & \text{in } \Omega, \quad \text{for } i \in \mathbb{N}, \\ p = p_a, \quad x \in \partial\Omega. & \end{cases} \tag{2.3}$$

Then

$$\|p_i - p_0\|_{L^q(\Omega)} \to 0, \quad \text{as} \quad i \to \infty,$$

for any $q < \infty$.

Proof. We first obtain a priori estimates in $H^1(\Omega)$ using $p_i - p_a$ as a test function in (2.3)

$$\int_\Omega (h_i^3 p_i + \lambda h_i^2)|\nabla p_i|^2 \leq \Lambda \int_\Omega h_i p_i |\nabla p_i|.$$

Since

$$h_i p_i |\nabla p_i| \leq h_i^3 p_i |\nabla p_i|^2 + \frac{1}{4h_i} p_i,$$

$$\frac{1}{4h_i} p_i = \frac{1}{4h_i}(p_i - p_a) + \frac{p_a}{4h_i}$$

and

$$\int_\Omega \frac{1}{4h_i}(p_i - p_a) \leq C_1(\Omega, \underline{h}) \|\nabla p\|_{L^2(\Omega)},$$

we have

$$\|\nabla p_i\|_{L^2(\Omega)} \leq \frac{1}{\underline{h}^2} \left(\frac{p_a}{2\underline{h}} + C_1^2(\Omega, \underline{h}) \right).$$

Since $p_i = p_a$ in $\partial\Omega$, we have that p_i is uniformly bounded in $H^1(\Omega)$, and therefore there exists a subsequence $p_{ij} \rightharpoonup p^*$ in the weak topology of $H^1(\Omega)$. By uniqueness of weak solutions of (2.3) (see [4]), we obtain that $p^* = p_0$. Besides, any other sequence or subsequence of p_i converges weakly to p_0 in $H^1(\Omega)$. Thanks to the Rellich-Kondrachov Theorem, we have that $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is a compact embedding for every $q < \infty$. This completes the proof. □

A technical lemma.

Now, we present a technical lemma concerning the positivity of the $V(\Omega)$-norm of $p - p_a$.

LEMMA 2.7. We denote by $p_{\eta,\alpha}$ the solution to (1.3) for a given $(\eta,\alpha)\in A$. Then for any $\eta>0$ we have that

$$\epsilon(\eta)=\frac{1}{\eta\Lambda}\int_\Omega (h^3p+\lambda h^2)\frac{|\nabla p_{\eta,\alpha}|^2}{p}>0.$$

Proof. First, we notice that for a given (η,α), the corresponding solution $p_{\eta,\alpha}$ satisfies

$$p_{\eta,\alpha}(\theta,x)=p_{\eta,\alpha-\gamma}(\theta-\gamma,x), \tag{2.4}$$

for any $\gamma\in(0,2\pi)$. This fact is a consequence of the symmetry of h, and the proof follows by direct substitution in (1.3) and the uniqueness of solutions of (1.3).

As a consequence of (2.4), we have that $\int_\Omega |\nabla p_{\eta,\alpha}|^2$ is independent of α. It is easy to check that for any $\eta>0$, $p_{\eta,\alpha}=p_a$ is not a solution to the problem. Therefore, thanks to the boundary conditions and property (2.4), we know that

$$\int_\Omega |\nabla p_{\eta,\alpha}|^2>0$$

and we have

$$\epsilon(\eta)=\frac{1}{\eta\Lambda}\int_\Omega (h^3p+\lambda h^2)\frac{|\nabla p_{\eta,\alpha}|^2}{p}>\frac{\min\{h^3\}}{\eta\Lambda}\int_\Omega|\nabla p_{\eta,\alpha}|^2=\frac{(1-\eta)^3}{\eta\Lambda}\int_\Omega|\nabla p_{\eta,\alpha}|^2>0,$$

and the proof ends. □

COROLLARY 2.8. There exists $\overline{\epsilon}\le\infty$ defined by

$$\overline{\epsilon}:=\sup_{\eta\in(0,1)}\epsilon(\eta) \tag{2.5}$$

such that $\overline{\epsilon}>0$.

3. Proof of Theorem 1.1. As in [5], the proof of the theorem is based on a fixed point argument; hence we introduce an auxiliary function $G=(G_1,G_2):=A\to\mathbb{R}^2$ defined by

$$G_1(\eta,\alpha):=\int_\Omega p_{\eta,\alpha}\cos\theta-F_1,\quad G_2(\eta,\alpha):=\int_\Omega p_{\eta,\alpha}\sin\theta-F_2, \tag{3.1}$$

where $p_{\eta,\alpha}$ is a solution to (1.3) for a given $(\eta,\alpha)\in A$. Notice that G represents the balances of forces when the inner cylinder is at a fixed position (η,α). The equilibrium solutions of the problem are defined by the zeros of G. As in [5] we compute the degree of G to obtain the existence of at least a zero of G which gives the existence of solutions to the inverse problem.

Notice that, since $\eta\in[0,1)$, h satisfies

$$0<1-\eta\le h\le 1+\eta\le 2, \tag{3.2}$$

which implies that for any fixed $(\eta,\alpha)\in A$ there exists a unique weak solution to (1.3) (see [4] for details). As a consequence of the uniqueness of the direct problem (for a given (η,α)), G is well defined, and thanks to Lemma 2.6, G is a continuous function in A.

By Theorem 3.4 and Corollary 3.6 in [4] we know that $p\in L^\infty(\Omega)$ and

$$p\ge C>0,$$

where C is a positive constant depending on Ω, p_a, λ, Λ and η but, in view of (2.4), independent of α. Now, we choose $\overline{\eta} < 1$ such that $\epsilon(\overline{\eta})$ defined in Lemma 2.7 satisfies

$$\epsilon(\overline{\eta}) > |F|. \tag{3.3}$$

Notice that by assumption (1.5) and Corollary 2.8 such $\overline{\eta}$ exists. Since G is not defined in ∂A, it is not possible to obtain the degree of G in A; hence we have to define a compact set where G is well defined. We define $A_{\overline{\eta}} \subset A$ as

$$A_{\overline{\eta}} := \Big\{(\eta_1,\eta_2) \in \mathbb{R}^2 : \quad \eta_1^2+\eta_2^2 \le \overline{\eta}\Big\} = \Big\{(\eta,\alpha) \in \mathbb{R}_+ \times [0,2\pi), \quad \eta \le \overline{\eta}\Big\}$$

and compute the degree of G in $A_{\overline{\eta}}$. We first consider the auxiliary field $W : A_{\overline{\eta}} \to \mathbb{R}^2$ defined by

$$W(\eta_1,\eta_2) = (-\eta_2,\eta_1) = (-\eta\sin\alpha, \eta\cos\alpha).$$

It is clear that by the linearity of W, $deg(W, A_{\overline{\eta}}, y_0) = 1$ for any $y_0 \in W(A_{\overline{\eta}}) = A_{\overline{\eta}}$. In order to apply Theorem 1.12 in [10] (see Theorem 2.4 in the previous section) we compute the product $G \cdot W^t$ in the circumference $\partial A_{\overline{\eta}}$:

$$G\cdot W^t = \overline{\eta}\left(-\int_\Omega p\cos\theta d\theta dx \sin\alpha + \int_\Omega p\sin\theta d\theta dx\cos\alpha + (F_2\cos\alpha - F_1\sin\alpha)\right).$$

Let us define $\vec{U} = (\Lambda, 0)$, take $\phi = \log\frac{p}{p_a}$ as a test function in (1.3) and integrate by parts to obtain

$$\int_\Omega (h^3p+\lambda h^2)\frac{|\nabla p|^2}{p} = \Lambda\int_\Omega h\frac{\partial p}{\partial\theta} = -\int_\Omega p\nabla h\vec{U} + \int_{\partial\Omega} hp_a\vec{U}\vec{n}.$$

Since h is independent of x and $p_a\vec{U}\vec{n}$ is constant in $x=0$ and $x=1$, we have

$$\int_{\partial\Omega} hp_a\vec{U}\vec{n} = p_aU_1\int_0^{2\pi} h(\theta,0) - p_aU_1\int_0^{2\pi} h(\theta,1) = 0$$

and therefore

$$\int_\Omega (h^3p+\lambda h^2)\frac{|\nabla p|^2}{p} = \int_\Omega h\vec{U}\nabla p = -\int_\Omega p\nabla h\vec{U} = -\Lambda\int_\Omega p\frac{\partial h}{\partial\theta}. \tag{3.4}$$

Notice that

$$-\Lambda\int_\Omega p\frac{\partial h}{\partial\theta} = \Lambda\int_\Omega p\overline{\eta}\sin(\theta-\alpha) = \Lambda\overline{\eta}\left[\left(\int_\Omega p\cos\theta, \int_\Omega p\sin\theta\right)\cdot(-\sin\alpha,\cos\alpha)^t\right].$$

Then,

$$\begin{aligned}\int_\Omega (h^3p+\lambda h^2)\frac{|\nabla p|^2}{p} &= \Lambda\overline{\eta}[(\int_\Omega p\cos\theta, \int_\Omega p\sin\theta)\cdot(-\sin\alpha,\cos\alpha)^t]\\ &= \Lambda\overline{\eta}(G+F)\cdot(-\sin\alpha,\cos\alpha)^t = \Lambda\overline{\eta}(G\cdot(-\sin\alpha,\cos\alpha)^t - F_1\sin\alpha + F_2\cos\alpha).\end{aligned}$$

Since

$$G\cdot(-\sin\alpha,\cos\alpha)^t = \frac{1}{\Lambda\overline{\eta}}\int_\Omega(\lambda h^2+h^3p)\frac{|\nabla p|^2}{p} + (F_1\sin\alpha - F_2\cos\alpha)$$

we have

$$G\cdot(-\sin\alpha,\cos\alpha)^t \ge \frac{1}{\Lambda\overline{\eta}}\int_\Omega(\lambda h^2+h^3p)\frac{|\nabla p|^2}{p} - |F| = \epsilon(\overline{\eta}) - |F|.$$

By Corollary 2.8 we have that, thanks to assumption (3.3),

$$G \cdot (-\sin\alpha, \cos\alpha) \geq \epsilon(\overline{\eta}) - |F| > 0,$$

which implies $G \cdot W^t > 0$ in $\partial A_{\overline{\eta}}$. By Theorem 2.4, both fields G and W have the same degree in $A_{\overline{\eta}}$. Since $deg(W, A_{\overline{\eta}}) = 1$ we have that

$$deg(G, A_{\overline{\eta}}) = 1,$$

which gives us the existence of at least a zero $(\eta^*, \alpha^*) \in A_{\overline{\eta}}$ of G. The proof of the theorem ends taking p^* as the unique solution to (1.3) with h defined in (1.2) for $(\eta, \alpha) = (\eta^*, \alpha^*)$.

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