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Asymptotic behavior and global existence of solutions to a two-species chemotaxis system with two chemicals

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Abstract. We study a nonlinear system of differential equations describing the evolution of a competitive two-species chemotaxis system with two chemicals in a bounded domain. The system consists of four PDEs, two equations of parabolic type describing the evolution of the competitive species and two elliptic equations modeling the distribution of the chemicals. By introducing global competitive/cooperative factors, we obtain, for different ranges of parameters, that any positive and bounded solution converges to a spatially homogeneous state. The proofs rely on the comparison principle for spatially homogeneous sub- and super-solutions. The existence and uniqueness of global classical solution are proved under assumptions on the initial data and appropriate conditions on the parameters of the system. Such solution stabilizes to spatially homogeneous equilibria in the large time limit, given coexistence or extinction of solutions for a range of parameters.

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1. Introduction

One of the mathematically challenging problems in the population dynamics is finding conditions under which all of the populations coexist. Since 1970s, chemotaxis has been studied from a mathematical point of view, starting with the first models of PDEs which were suggested by Keller and Segel [10,11], the mathematical bibliography is extensive, more details about obtained results in the area during the last decades can be found in [1,6,7,12,14–16]. To model the movement of several species according to the mechanism of chemotaxis, the classical Keller–Segel model was extended to some multi-species case. We consider a system describing the evolution of two competitive species u_1 and u_2 with chemotactic movement. One of the first mathematical models in population dynamics is the well known Lotka–Volterra system (see [13] and [27]), the system consists of two ordinary differential equations describing the evolution of a predator u_1 and a prey u_2, of the form

$$u_{1t} = g_1(u_1, u_2), \qquad u_{2t} = g_2(u_1, u_2).$$

If the species diffuse, the system describing their concentrations becomes a reaction diffusion system, already considered by Turing [26] in 1952 for the linear case and widely studied after him. There also exist common examples in nature, where the biological species detect the other species by the chemical secreted by the second one, moving toward a higher concentration of the chemical (chemotaxis) or away from it (chemorepulsion). Different types of situation may occur depending of the chemotactic coefficients and the competitive terms given by g_i.

In this article, two different aspects of the interaction are considered:

- The competition between the species is presented by the functions g_i, for $i = 1, 2$ whose expressions are the following

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$$g_1(u_1,u_2) = \mu_1 u_1(1-u_1-a_1u_2),$$
$$g_2(u_1,u_2) = \mu_2 u_2(1-a_2u_1-u_2),$$

where the coefficients a_i and μ_i (for $i=1,2$) are positive given data, assumed constants.

- "The chemotactic movement" of the species is due to chemical substances secreted by the other species. These type of interactions are determined by the sign of the chemotactic coefficients, which can be positive or negative, moving toward higher or lower concentrations of the substance secreted by the other species which is directly related to its density.

Considering the sign of the chemotactic coefficient, we have the following classification of the system:

1. $\chi_i = 0$, $i=1,2$. The biological species do not interact via chemical substances.
2. $\chi_i \neq 0, i=1,2$.
   (a) $\chi_i > 0$, $i=1,2$. Even there is competition between the biological species, they approach each other moving toward a higher concentration of the chemical secreted by the other species.
   (b) $\chi_i < 0$, $i=1,2$. The species are mutually repelled, so both u_i, $i=1,2$ move toward the lower concentration of the chemical secreted by the other species.
   (c) $\chi_1 > 0$, $\chi_2 < 0$ or $\chi_1 > 0$, $\chi_2 < 0$. One of the species moves toward the higher concentration of the chemical secreted by the other species, while the second one moves to the lower concentration of the chemical secreted by the other one.
3. $\chi_1 \neq 0, \chi_2 = 0$. The species u_2 does not present chemotactic movement, while the movement of the species u_1 is oriented by the chemical secreted by u_2. In this case, the system is reduced to a system of three equation where two types of systems appear:
   (a) $\chi_1 > 0$, $\chi_2 = 0$. u_1 moves toward the higher concentration of the chemical secreted by u_2 which is directly related with the higher concentration of u_2.
   (b) $\chi_1 < 0$, $\chi_2 = 0$. u_1 moves toward low concentrations of the chemical secreted by u_2 trying to avoid the other species.

For the population densities u_1, u_2 and the concentrations of the chemoattractants v_1, v_2, the classical models lead to

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \cdot \nabla v_1) + g_1(u_1,u_2), & x \in \Omega, \quad t>0, \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \cdot \nabla v_2) + g_2(u_1,u_2), & x \in \Omega, \quad t>0, \\ -\Delta v_1 + \alpha_1 v_1 = \beta_1 u_2, & x \in \Omega, \quad t>0, \\ -\Delta v_2 + \alpha_2 v_2 = \beta_2 u_1, & x \in \Omega, \quad t>0, \end{cases} \tag{1.1}$$

where $d_i > 0$ are the diffusion coefficients, χ_i are the chemorepulsion coefficients, $\alpha_i > 0$, $\beta_i > 0$ for $i=1,2$ and Ω is a bounded, open regular domain of $\mathbb{R}^n$, for $n \geq 1$ with regular boundary. We assume Neumann boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v_1}{\partial \nu} = \frac{\partial v_2}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t>0, \tag{1.2}$$

and bounded initial data

$$u_1(0,x) = u_1^0(x), \qquad u_2(0,x) = u_2^0(x), \quad x \in \Omega, \tag{1.3}$$

satisfying

$$u_i^0(x) \in C^{2+\alpha}(\Omega), \quad \frac{\partial u_i^0}{\partial \nu} = 0 \quad \text{in } \partial\Omega, \quad \underline{u}_i^0 \leq u_i^0(x) \;\; x \in \Omega, \tag{1.4}$$

for some $\alpha > 0$ and $\underline{u}_i^0 > 0$ for $i = 1, 2,$. Throughout the present article, we consider the system under the restrictions

$$2|\chi_1|\beta_1 + \mu_1 a_1 < \mu_2, \tag{1.5}$$

$$2|\chi_2|\beta_2 + \mu_2 a_2 < \mu_1, \tag{1.6}$$

for positive coefficients a_i, for $i = 1, 2$ and $\chi_i \in \mathbb{R}$.

Assumptions (1.5) and (1.6) restrict the study to the case where the reaction terms are dominant in comparison with the chemotactic terms.

The main results of the paper describing the global existence of solutions of (1.1)–(1.3) and its asymptotic behavior are enclosed in the following theorem.

Theorem 1.1. *Under assumptions (1.4)–(1.6), for any nonnegative initial data (u_1^0, u_2^0) as in (1.4), there exists a unique solution to (1.1)–(1.3) satisfying*

$$u_i, v_i \in C_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_T), \qquad \text{for } i = 1, 2 \text{ and any } T < \infty;$$

moreover, the solution (u_1, u_2, v_1, v_2) of (1.1) fulfills

$$\lim_{t\to\infty} \|u_i - u_i^*\|_{L^\infty(\Omega)} + \|v_j - u_i^*\|_{L^\infty(\Omega)} = 0, \quad i, j = 1, 2, \quad i \neq j,$$

where u_i^* are given by

1.
$$u_1^* = 1 \qquad \text{and} \qquad u_2^* = 0, \tag{1.7}$$

if relations

$$a_1 < 1, \tag{1.8}$$

$$a_2 > 1 \tag{1.9}$$

hold, and

2.
$$u_1^* = \frac{1 - a_1}{1 - a_1 a_2} \qquad \text{and} \qquad u_2^* = \frac{1 - a_2}{1 - a_1 a_2}, \tag{1.10}$$

if (1.8) holds and relation (1.9) is replaced by

$$a_2 < 1. \tag{1.11}$$

Theorem 1.1 describes the behavior of a competitive system with two biological species with chemotactic movement. In the first case, the species u_1 persists and its density converges to an homogeneous spatial distribution, while species u_2 vanishes as t goes to infinity. In the second case, where $a_1 < 1$ and $a_2 < 1$, both species persist and the densities stabilize in some constant steady state given by (1.10).

In order to prove the asymptotic behavior of solutions, we introduce an auxiliary system of ODEs. The proof is based on the comparison principle, in other words, we need a relation of order to bound the solution of problem (1.1), (u_1, u_2, v_1, v_2), between the solutions of an ODE's system. The moving rectangles method is adapted to prove the convergence to a constant steady state for some range of parameters and initial data. The auxiliary system of ODE is presented in Sect. 2, where the qualitative properties of the system are studied. The comparison between the solutions of the ODEs system and the solutions of the PDEs is presented in Sect. 3; finally, in the last section the previous results are applied to prove the existence of global solutions and their asymptotic behavior.

Stability and asymptotic behavior of chemotactic systems with two biological species have been already studied in Tello and Winkler [25], where the stability of homogeneous steady states is obtained for one chemical (see also Stinner et al. [23] for the competitive exclusion case and Bai and Winkler [4] and Blacket al. [3] for the fully parabolic system).

In Zhang et al. [29], system (1.1) is studied under the assumption $\mu_1\mu_2 > \chi_1\chi_2$ for $\chi_i \geq 0$, for $a_1, a_2 \in (0,1)$. The authors prove the global existence of solutions for any bounded and regular initial data. If, in addition, $\chi_i < a_i\mu_i$ the convergence to the constant steady state

$$\left(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{1-a_1}{1-a_1a_2}\right)$$

is obtained.

Recently, Zhang [28] has studied the competitive case under assumptions

$$a_1 > 1 > a_2 > 0,$$

for $\mu_1\mu_2 > \chi_1\chi_2$ and $\chi_1 \leq a_1\mu_1$, $\chi_2 < \mu_2$. The author proves the global existence of solutions and the convergence of (u_1, u_2) to the homogeneous steady state $(0,1)$. Notice that, assumptions (1.5), (1.6) consider a different range of parameters than the studied in [29] and [28] where only positive parameters χ_i are considered. The problem remains open for large χ_i which is not considered in [28,29] neither in this article. In particular if $g_1 = g_2 = 0$, blow up occurs for a range of initial data as it is shown in Tao and Winkler [24].

Recently, Zheng et al. [30] have studied the fully parabolic system for $g_1 = g_2 = 0$, i.e., the case, where there is no competition between the species. The authors obtained the global existence of solutions and the coexistence of the species for $\chi_i \in (-1,1)$.

The fully parabolic system is also studied in Black [2], where the author considers the following three situations:

1. weak competition case $a_2, a_1 \in (0,1)$,
2. partially strong competition case $a_2 < 1 < a_1$,
3. fully strong competition case $a_2, a_1 > 1$.

The author has obtained the global existence of solutions under assumption

$$\mu_i > \frac{7}{2}\chi_i^2.$$

Moreover, coexistence occurs for case 1 for a range of parameters and a constant steady state is asymptotically stable. While in case 2, extinction of one species occurs, and the semi-trivial solution is globally asymptotically stable. In case 3, there exist two solutions which are locally asymptotically stable, the first one are given by $u_1 = 0$ and $v_2 = 0$ and u_2 and v_1 given by positive constants. The second solution is the symmetric case, i.e., $u_2 = 0$ and $v_1 = 0$, for $u_1 > 0$ and $v_2 > 0$.

2. Associated ODE system and qualitative properties

In this section, we study an auxiliary system of ordinary differential equations related to the original nonlinear system (1.1) to obtain the asymptotic behavior of its solutions.

Developing the second-order terms in (1.1), we get

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 - \chi_1\nabla u_1\cdot\nabla v_1 + \chi_1 u_1\left(\beta_1 u_2 - \alpha_1 v_1\right) + g_1, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 - \chi_2\nabla u_2\cdot\nabla v_2 + \chi_2 u_2\left(\beta_2 u_1 - \alpha_2 v_2\right) + g_2. \end{cases} \tag{2.1}$$

Let us consider $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2) = (\overline{u}_1(t), \underline{u}_1(t), \overline{u}_2(t), \underline{u}_2(t))$ the solution of the initial value problem

$$\begin{cases} \overline{u}_1' = |\chi_1|\beta_1\overline{u}_1(\overline{u}_2 - \underline{u}_2) + \mu_1\overline{u}_1(1 - \overline{u}_1 - a_1\underline{u}_2), \\ \underline{u}_1' = |\chi_1|\beta_1\underline{u}_1(\underline{u}_2 - \overline{u}_2) + \mu_1\underline{u}_1(1 - \underline{u}_1 - a_1\overline{u}_2), \\ \overline{u}_2' = |\chi_2|\beta_2\overline{u}_2(\overline{u}_1 - \underline{u}_1) + \mu_2\overline{u}_2(1 - a_2\underline{u}_1 - \overline{u}_2), \\ \underline{u}_2' = |\chi_2|\beta_2\underline{u}_2(\underline{u}_1 - \overline{u}_1) + \mu_2\underline{u}_2(1 - a_2\overline{u}_1 - \underline{u}_2), \end{cases} \tag{2.2}$$

for $t \in (0, \infty)$, with initial data

$$\overline{u}_1(0) = \overline{u}_1^0, \quad \underline{u}_1(0) = \underline{u}_1^0, \quad \overline{u}_2(0) = \overline{u}_2^0, \quad \underline{u}_2(0) = \underline{u}_2^0, \tag{2.3}$$

satisfying

$$0 < \underline{u}_1^0 < \overline{u}_1^0 \quad \text{and} \quad 0 < \underline{u}_2^0 < \overline{u}_2^0. \tag{2.4}$$

Notice that, standard ODE theory gives the local existence of solutions of systems (2.2)–(2.3).

Throughout this section, we prove that the two pairs of solutions of the ODE's system (2.2), i.e., $(\overline{u}_1, \underline{u}_1)$ and $(\overline{u}_2, \underline{u}_2)$ have the same constant limit u_1^* and u_2^*, respectively, and, hence, also any function between them. Recall that we are working under hypothesis (1.5)–(1.6), i.e.,

$$2|\chi_1|\beta_1 + \mu_1 a_1 < \mu_2 \quad \text{and} \qquad 2|\chi_2|\beta_2 + \mu_2 a_2 < \mu_1.$$

The stationary states of (2.2) satisfying $\underline{u}_1 = \overline{u}_1$, $\underline{u}_2 = \overline{u}_2$, are the trivial steady state

$$(0, 0, 0, 0),$$

the semi-trivial steady states

$$(0, 0, 1, 1) \tag{2.5}$$

and

$$(1, 1, 0, 0) \tag{2.6}$$

and a fourth steady state given by

$$\left(\frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2} \right). \tag{2.7}$$

Now, we study the properties of the solutions of the above system, i.e., we find a relationship between solutions $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ when the initial data $(\overline{u}_1^0, \underline{u}_1^0, \overline{u}_2^0, \underline{u}_2^0)$ satisfies (2.4), i.e., the initial ordering is inherited by the solution. Furthermore, we prove that $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ are actually global in time and bounded. The constant steady states can be both positive or one positive and one negative.

2.1. Case $a_1 < 1$ and $a_2 > 1$

In this section, we focus on the case of competitive exclusion of species, i.e.,

$$a_1 < 1 \qquad a_2 > 1.$$

Notice that under assumptions (1.8)–(1.9), the fourth steady state has two positive coordinates and two negative, since the solution remains nonnegative, provided the boundedness an convergence of the solutions, at least two solutions tend to zero despite of the initial data.

We denote the u_1^* and u_2^* the semi-trivial steady state given by (1.7), i.e.,

$$u_1^* = 1, \qquad u_2^* = 0.$$

We proceed to prove the asymptotic behavior of the solutions; the result is enclosed in the following theorem.

Theorem 2.1. *Let $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ be the solutions of system (2.2) and (u_1^*, u_2^*) given by (1.7). Under assumptions (1.5)–(1.6) and (1.8)–(1.9) the below limits hold*

$$\begin{aligned} \overline{u}_1(t) &\to u_1^* \quad & \underline{u}_1(t) &\to u_1^* \quad & t &\to \infty \\ \overline{u}_2(t) &\to u_2^* \quad & \underline{u}_2(t) &\to u_2^* \quad & t &\to \infty. \end{aligned}$$

In order to demonstrate Theorem 2.1, we prove that $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ are actually global in time and bounded.

Lemma 2.1. *The solution of system* (2.2)*, with initial data* (2.3) *verifying* (2.4)*, satisfies*

$$0 < \underline{u}_1 < \overline{u}_1 \tag{2.8}$$

$$0 < \underline{u}_2 < \overline{u}_2 \tag{2.9}$$

for $t \in (0, \infty)$.

Proof. We first see that $\underline{u}_i > 0$, for $i = 1, 2$. We can write the corresponding equations of $\underline{u}_i$ from system (2.2) as $\underline{u}_i' = \underline{u}_i f_i(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$, f_i being smooth functions. One check easily that $\underline{u}_i = 0$ is a solution of the previous equation. By existence and uniqueness of the solution and taking into account that the initial data are positive $\underline{u}_i^0 > 0$ it follows that $\underline{u}_i(t) > 0$ for all $t > 0$.

We prove $\underline{u}_i < \overline{u}_i$ arguing by contradiction. Suppose (2.8) and (2.9) are false, i.e., there exists t_0 in $(0, T_{\max})$ such that $\underline{u}_1(t) < \overline{u}_1(t)$ and $\underline{u}_2(t) < \overline{u}_2(t)$ for all $t \in (0, t_0)$ and one of the following cases occurs

$$\underline{u}_1(t_0) = \overline{u}_1(t_0) \quad \underline{u}_2(t_0) < \overline{u}_2(t_0), \tag{2.10}$$

$$\underline{u}_1(t_0) < \overline{u}_1(t_0) \quad \underline{u}_2(t_0) = \overline{u}_2(t_0), \tag{2.11}$$

$$\underline{u}_1(t_0) = \overline{u}_1(t_0) \quad \underline{u}_2(t_0) = \overline{u}_2(t_0). \tag{2.12}$$

If case (2.10) occurs, we divide the first and second equations in (2.2) by $\overline{u}_1$ and $\underline{u}_1$, respectively, and subtracting them we get

$$\frac{\overline{u}_1'}{\overline{u}_1} - \frac{\underline{u}_1'}{\underline{u}_1} = (2|\chi_1|\beta_1 + \mu_1 a_1)(\overline{u}_2 - \underline{u}_2) - \mu_1(\overline{u}_1 - \underline{u}_1).$$

By continuity of the functions involved in the previous equation and by (2.10), we can find ϵ such that $\overline{u}_1 - \underline{u}_1 < \epsilon$ and $(2|\chi_1|\beta_1 + \mu_1 a_1)(\overline{u}_2 - \underline{u}_2) > \mu_1\epsilon$, for a certain δ small enough, so, these inequalities hold in the interval $(t_0 - \delta, t_0)$. Then, in $(t_0 - \delta, t_0)$ we have

$$\frac{\overline{u}_1'}{\overline{u}_1} - \frac{\underline{u}_1'}{\underline{u}_1} > 0$$

and integrating over $(t_0 - \delta, t_0)$ we obtain

$$\ln \overline{u}_1(t_0) - \ln \underline{u}_1(t_0) > 0,$$

which contradicts (2.10). Case (2.11) is similar to the previous case; therefore, we omit the details.

To prove case (2.12), we introduce the functions f and g defined by

$$f = \overline{u}_1 - \underline{u}_1,$$
$$g = \overline{u}_2 - \underline{u}_2.$$

We have the following differential equation for f

$$f' = |\chi_1|\beta_1(\overline{u}_2 - \underline{u}_2)(\overline{u}_1 + \underline{u}_1) + \mu_1(\overline{u}_1 - \underline{u}_1) - \mu_1(\overline{u}_1^2 - \underline{u}_1^2) - \mu_1 a_1(\overline{u}_1\underline{u}_2 - \underline{u}_1\overline{u}_2),$$

which can be rewritten as

$$f' = h_1\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) f + h_2\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) g.$$

for

$$h_1\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) := \mu_1 - \mu_1(\overline{u}_1 + \underline{u}_1) - \frac{1}{2}\mu_1 a_1(\overline{u}_2 + \underline{u}_2)$$

and

$$h_2\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) := |\chi_1|\beta_1(\overline{u}_1 + \underline{u}_1) + \frac{1}{2}\mu_1 a_1(\overline{u}_1 + \underline{u}_1).$$

Proceeding the same way on g, we get

$$g' = h_3\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) f + h_4\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) g,$$

for

$$h_3\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) := |\chi_2|\beta_2(\overline{u}_2 + \underline{u}_2) + \frac{1}{2}\mu_2 a_2(\overline{u}_2 + \underline{u}_2),$$

and

$$h_4\left(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2\right) := \mu_2 - \mu_2(\overline{u}_2 + \underline{u}_2) - \frac{1}{2}\mu_2 a_2(\overline{u}_1 + \underline{u}_1).$$

We turn our attention to the system

$$\begin{cases} f' = h_1 \cdot f + h_2 \cdot g, \\ g' = h_3 \cdot f + h_4 \cdot g, \\ f(t_0) = 0 \quad g(t_0) = 0, \end{cases}$$

where it can be easily checked that $(f, g) = (0, 0)$ is a solution of the system and by uniqueness of solution together with the initial data, we conclude that $f = 0$, $g = 0$ in the interval $[0, t_0)$, which contradicts the definition of t_0 and ends the proof. □

Lemma 2.2. *Let* (1.5), (1.6) *hold. Then, the solution of the system* (2.2) *satisfies*

$$\underline{u}_1(t) \le 1, \quad \underline{u}_2(t) \le 1 \qquad \textit{for} \quad t \in (0, \infty), \tag{2.13}$$

provided

$$\underline{u}_1(0) \le 1, \quad \underline{u}_2(0) \le 1.$$

Proof. Dividing the second equation in (2.2) by $\underline{u}_1$, we see that

$$\frac{\underline{u}_1'}{\underline{u}_1} \le \mu_1 - \mu_1 \underline{u}_1. \tag{2.14}$$

So $\underline{u}_1(t)$ is a subsolution of the ODE $y_1' = \mu_1(y_1 - y^2)$ which can be solved easily and the solution is

$$y_1 := \frac{1}{c_1 e^{-\mu_1 t} + 1} \tag{2.15}$$

for $c_1 := \underline{u}_1^{-1}(0) - 1$. Thus $\underline{u}_1$ satisfies

$$\underline{u}_1 \le \frac{1}{c e^{-\mu_1 t} + 1} \le 1 := u_1^*. \tag{2.16}$$

In the same way, we prove

$$\underline{u}_2(t) \le y_2(t) \le 1,$$

and the proof ends. □

Lemma 2.3. *Under assumptions* (1.5)–(1.6) *there exists a positive constant $C > 0$ such that the solution* $(\overline{u}_1, \overline{u}_2)$ *of* (2.2) *satisfies*

$$\overline{u}_1 \overline{u}_2 \le C \quad \textit{for} \quad t \in (0, \infty).$$

Proof. Let $\phi(t)$ be the function defined as $\phi(t) = \ln \overline{u}_1(t) + \ln \overline{u}_2(t)$. Dividing the first equation in (2.2) by $\overline{u}_1$ and the third one by $\overline{u}_2$, after adding up, we have

$$\phi' = \frac{\overline{u}_1'}{\overline{u}_1} + \frac{\overline{u}_2'}{\overline{u}_2} \le \mu_1 + \mu_2 + (|\chi_2|\beta_2 - \mu_1)\overline{u}_1 + (|\chi_1|\beta_1 - \mu_2)\overline{u}_2.$$

Denoting $c_1 = \mu_1 + \mu_2 > 0$, $c_2 = \min\{\mu_1 - |\chi_2|\beta_2, \mu_2 - |\chi_1|\beta_1\} > 0$, in the above inequality, we claim

$$\phi'(t) \le c_1 - c_2(\overline{u}_1 + \overline{u}_2). \tag{2.17}$$

By mean of the inequality $\overline{u}_1 + \overline{u}_2 \geq 2\sqrt{\overline{u}_1\overline{u}_2} = 2e^{\frac{1}{2}\phi}$, we get that ϕ verifies the inequality

$$\phi'(t) \leq c_1 - c_2 2e^{\frac{1}{2}\phi}. \tag{2.18}$$

Thus, ϕ is a sub-solution of the equation

$$z' = c_1 - c_2 2e^{\frac{1}{2}z}, \tag{2.19}$$

with initial $z(0) := \ln \overline{u}_1^0 + \ln \overline{u}_2^0$. This equation has as stationary state $2\ln\frac{c_1}{2c_2}$, so $z \leq \max\left\{z^0, 2\ln\frac{c_1}{2c_2}\right\}$. Because of $\phi(t) \leq z(t)$ for all $t > 0$, we found an upper bound for $\ln\overline{u}_1 + \ln\overline{u}_2$ and we conclude that there exists a positive constant $C > 0$ such that $\overline{u}_1\overline{u}_2 \leq C$. □

Lemma 2.4. *Under hypothesis* (1.5)–(1.6)*, there exists a positive constant* $K > 0$*, such that the solution* $(\overline{u}_1, \overline{u}_2)$ *of* (2.2) *satisfies for* $t \in (0, \infty)$

$$\overline{u}_1 \leq K, \quad \overline{u}_2 \leq K.$$

Proof. Taking the first equation of system (2.2) and by Lemma 2.3

$$\begin{aligned}
\overline{u}_1' &= \overline{u}_1[|\chi_1|\beta_1(\overline{u}_2 - \underline{u}_2) + \mu_1 - \mu_1\overline{u}_1 - \mu_1 a_1 \underline{u}_2] \\
&= |\chi_1|\beta_1\overline{u}_1\overline{u}_2 - |\chi_1|\beta_1\overline{u}_1\underline{u}_2 + \mu_1\overline{u}_1 - \mu_1\overline{u}_1^2 - \mu_1 a_1\overline{u}_1\underline{u}_2 \\
&\leq |\chi_1|\beta_1\overline{u}_1\overline{u}_2 + \mu_1\overline{u}_1 - \mu_1\overline{u}_1^2 \\
&\leq |\chi_1|\beta_1 C + \mu_1\overline{u}_1 - \mu_1\overline{u}_1^2.
\end{aligned}$$

By comparison, the previous inequality proves

$$\overline{u}_1 \leq \max\left\{\overline{u}_0, \frac{\mu_1 + \sqrt{\mu_1^2 + 4|\chi_1|\beta_1 C\mu_1}}{2\mu_1}\right\}.$$

In an analogous way, we find a positive constant K_2 such that $\overline{u}_2 \leq K_2$. Taking now $K = \max\{K_1, K_2\}$ the proof of Lemma 2.4 is complete. □

The following lemma will be used to prove that the super- and sub- solutions $\overline{u}_2$ and $\underline{u}_2$ are comparable in both directions:

Lemma 2.5. *Let* (1.5)–(1.6) *hold. Then, there exists a positive constant* $M > 0$ *such that*

$$\overline{u}_2 \leq M\underline{u}_2.$$

Proof. From equations (2.2):

$$\frac{\overline{u}_1'}{\overline{u}_1} - \frac{\underline{u}_1'}{\underline{u}_1} + \frac{\overline{u}_2'}{\overline{u}_2} - \frac{\underline{u}_2'}{\underline{u}_2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\overline{u}_1}{\underline{u}_1} + \ln\frac{\overline{u}_2}{\underline{u}_2}\right)$$

$$= A_1(\overline{u}_1 - \underline{u}_1) + A_2(\overline{u}_2 - \underline{u}_2),$$

where we denote by $A_1 := 2|\chi_2|\beta_2 + \mu_2 a_2 - \mu_1$ and $A_2 := 2|\chi_1|\beta_1 + \mu_1 a_1 - \mu_2$. By hypothesis (1.5), A_i are negative for $i = 1, 2$, i.e., $A_1 < 0$, and $A_2 < 0$. Taking $\epsilon = \min\{-A_1, -A_2\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\overline{u}_1}{\underline{u}_1} + \ln\frac{\overline{u}_2}{\underline{u}_2}\right) \leq -\epsilon[(\overline{u}_1 - \underline{u}_1) + (\overline{u}_2 - \underline{u}_2)] \leq 0. \tag{2.20}$$

Integrating this expression:

$$\left(\ln\frac{\overline{u}_1}{\underline{u}_1} + \ln\frac{\overline{u}_2}{\underline{u}_2}\right) \leq c_0,$$

for some positive constant $c_0 > 0$. Then, by the positivity of $\ln(\overline{u}_1/\underline{u}_1)$, because $\overline{u}_1 > \underline{u}_1$, it follows

$$\ln \frac{\overline{u}_2}{\underline{u}_2} \leq c_0 \Rightarrow \overline{u}_2 \leq M \underline{u}_2,$$

with $M = e^{c_0}$. □

Lemma 2.6. *Under assumptions* (1.5) *and* (1.6), *we have*

$$\overline{u}_i(t) - \underline{u}_i(t) \to 0 \quad \textit{as} \quad t \to \infty, \qquad \textit{and} \qquad \int_0^\infty (\overline{u}_i(t) - \underline{u}_i(t))dt < C.$$

Proof. We define the function

$$k(t) := \overline{u}_1 - \underline{u}_1.$$

Notice that, by Lemma 2.1 we have that $k > 0$ for any $t > 0$. Thanks to Lemma 2.4, we have that

$$|k'| = |\overline{u}_1' - \underline{u}_1'| \leq k_1 < \infty \tag{2.21}$$

where k_1 is independent of t.

Integrating in (2.20) over $(0, T)$, we get

$$\ln \frac{\overline{u}_1(T)}{\underline{u}_1(T)} + \ln \frac{\overline{u}_2(T)}{\underline{u}_2(T)} + \epsilon \int_0^T (\overline{u}_1 - \underline{u}_1)dt + \epsilon \int_0^T (\overline{u}_2 - \underline{u}_2)dt \leq \ln \frac{\overline{u}_1(0)}{\underline{u}_1(0)} + \ln \frac{\overline{u}_2(0)}{\underline{u}_2(0)}$$

and therefore, by Lemma 2.1 we have

$$\int_0^T (\overline{u}_1 - \underline{u}_1)dt \leq \frac{1}{\epsilon} \ln \frac{\overline{u}_1(0)}{\underline{u}_1(0)} + \frac{1}{\epsilon} \ln \frac{\overline{u}_2(0)}{\underline{u}_2(0)}.$$

We take limits when $T \to \infty$ to obtain

$$\int_0^\infty k(t)dt = \int_0^\infty (\overline{u}_1 - \underline{u}_1)dt \leq k_2 < \infty. \tag{2.22}$$

Relations (2.21)–(2.22) and Lemma 5.1 in Friedman–Tello [5] end the proof for $i = 1$.

The case $i = 2$ is proven in the same fashion. □

Lemma 2.7. *Let us consider that* (1.5) *and* (1.6) *are verified. For every positive coefficients a_1, a_2 verifying* (1.8) *and* (1.9), *the following statement holds*

$$\ln \underline{u}_2 + \mu_2(a_2 - 1) \int_0^t \overline{u}_1 + \mu_2(1 - a_1) \int_0^t \underline{u}_2 \leq C, \tag{2.23}$$

where C is independent of t.

Proof. Dividing the first and last equations in (2.2) by $\overline{u}_1$ and $\underline{u}_2$, respectively, multiplying the obtained equations by μ_2 and μ_1, respectively, and resting them, we see that

$$\begin{aligned}\mu_2 \frac{\mathrm{d}}{\mathrm{d}t} \ln \overline{u}_1 - \mu_1 \frac{\mathrm{d}}{\mathrm{d}t} \ln \underline{u}_2 = A_1(\overline{u}_1 - \underline{u}_1) + A_2(\overline{u}_2 - \underline{u}_2)& \\ + \mu_1\mu_2 \left[(a_2 - 1)\overline{u}_1 + (1 - a_1)\underline{u}_2\right],&\end{aligned}$$

where $A_1 = \mu_1|\chi_2|\beta_2 > 0$ and $A_2 = \mu_2|\chi_1|\beta_1 > 0$; after integration, and thanks to Lemma 2.6 we have

$$\mu_1 \ln \underline{u}_2 + \mu_1\mu_2(a_2 - 1)\int_0^t \overline{u}_1 + \mu_1\mu_2(1 - a_1)\int_0^t \underline{u}_2 \leq K + \mu_2 \ln \overline{u}_1. \tag{2.24}$$

Since $\ln \overline{u}_1 < C$, by Lemma 2.4 we conclude the result. □

Lemma 2.8. *We have*

$$\max\{\mu_1, \mu_2\}\int_0^t (\overline{u}_1 + \underline{u}_2) \geq \mu_1 t + C_1, \tag{2.25}$$

where C_1 is a constant independent of t.

Proof. From equation (2.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln \overline{u}_1 = |\chi_1|\beta_1(\overline{u}_2 - \underline{u}_2) + \mu_1 - \mu_1\overline{u}_1 - \mu_1 a_1 \underline{u}_2. \tag{2.26}$$

After integration in (2.26) and thanks to Lemmas 2.4 and 2.2, we conclude

$$\max\{\mu_1, \mu_2\}\int_0^t (\overline{u}_1 + \underline{u}_2) \geq \mu_1 t + C_1.$$

□

Lemma 2.9. *Let (1.5) and (1.6) hold. For every positive coefficients a_1, a_2 verifying relations (1.8) and (1.9), we have*

$$\underline{u}_2 \leq c_1 e^{-c_2 t},$$

which implies

$$\underline{u}_2 \to 0, \quad \textit{as} \quad t \to \infty. \tag{2.27}$$

Proof. The proof is a combination of previous lemmas, i.e., by relations (2.23) and (2.25) we get

$$K_1 t + K_2 \leq \int_0^t \overline{u}_1 + \int_0^t \underline{u}_2 \leq K_3 - K_4 \ln \underline{u}_2 \tag{2.28}$$

with K_i independent of t, K_1 and K_4 positive. Thus, $\underline{u}_2$ satisfies

$$\ln \underline{u}_2 \leq k_1 - k_2 t$$

which implies

$$\underline{u}_2 \leq c_1 e^{-c_2 t}.$$

Taking limits when $t \to \infty$, the proof ends. □

Lemma 2.10. *The following statement is true for $\overline{u}_1$ and $\underline{u}_2$ solutions of (2.2) if (1.5) and (1.6) hold*

$$\liminf_{t\to\infty} \overline{u}_1(t) \geq u_1^*.$$

Proof. Thanks to the previous Lemma, we have that for any $\epsilon > 0$ there exists $T_\epsilon < \infty$ such that $\underline{u}_2(t) \leq \epsilon$, $\forall t > T_\epsilon$ and we have in the first equation in (2.2), the first term on the right side is positive due to $\overline{u}_2 > \underline{u}_2$, that is

$$\overline{u}_1' \geq -\mu_1 \overline{u}_1^2 + \mu_1 \overline{u}_1 - \mu_1 a_1 \epsilon \overline{u}_1.$$

Thus, $\overline{u}_1$ is a super solution of the ordinary differential equation

$$y' = -\mu_1 y^2 + \mu_1(1 - a_1\epsilon)y,$$

whose solution is

$$y_\epsilon(t) = \frac{1}{\frac{1}{1-a_1\epsilon} + k_\epsilon e^{-\mu_1(1-a_1\epsilon)t}}, \quad \text{for} \quad k_\epsilon = \frac{1}{\overline{u}_1(T_\epsilon)} - \frac{1}{1-a_1\epsilon} \tag{2.29}$$

such that

$$y_\epsilon(t) \leq \overline{u}_1(t) \quad \text{for} \quad t \in (T_\epsilon, \infty).$$

Back to the solution $\overline{u}_1$, we conclude

$$\liminf_{t\to\infty} \overline{u}_1 \geq \lim_{t\to\infty} y_\epsilon = 1 - a_1\epsilon$$

We take limits when $\epsilon \to 0$ to end the proof. □

End of the proof of Theorem 2.1. Thanks to Lemmas 2.1 and 2.5, we conclude

$$\lim_{t\to\infty} \overline{u}_2 = \lim_{t\to\infty} \underline{u}_2 = 0.$$

As a consequence of Lemmas 2.2 and 2.10, we have

$$\liminf_{t\to\infty} \underline{u}_1 \leq u_1^* \leq \limsup_{t\to\infty} \overline{u}_1.$$

Moreover, by Lemma 2.6, we conclude the proof of Theorem 2.1. □

2.2. Case $a_1 < 1$ and $a_2 < 1$

In this section, we consider the following assumption

$$a_2 < 1,$$

instead of (1.9). We prove that there exists a unique globally stable steady with positive coordinates which corresponds to the coexistence of the species. The following theorem gives a precise description of the asymptotic behavior of the solutions of system (2.2).

Theorem 2.2. *Let $(\overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)$ be the solutions of system (2.2) and (u_1^*, u_2^*) given by (2.7), i.e.,*

$$u_1^* = \frac{1-a_1}{1-a_1a_2} \quad \text{and} \quad u_2^* = \frac{1-a_2}{1-a_2a_1}.$$

Under assumptions (1.5)–(1.6), (1.8) and (1.11) the below limits hold

$$\overline{u}_1(t) \to u_1^*, \quad \underline{u}_1(t) \to u_1^*, \quad \overline{u}_2(t) \to u_2^*, \quad \underline{u}_2(t) \to u_2^* \quad as \quad t \to \infty.$$

The importance of this result lies in obtaining $(\overline{u}_i - \underline{u}_i) \to 0$ when $t \to \infty$, $i = 1, 2$, which implies that any other function bounded between them will inherit their asymptotic behavior. To demonstrate the theorem, we consider the following steps.

Step 1.

$$0 < \underline{u}_1 < \overline{u}_1, \qquad 0 < \underline{u}_2 < \overline{u}_2.$$

Step 2.

$$\underline{u}_1 \le \underline{u}_1^* = \overline{u}_1^* \le \overline{u}_1, \qquad \underline{u}_2 \le \underline{u}_2^* = \overline{u}_2^* \le \overline{u}_2,$$

for all $t \in (0,\infty)$.

Step 3. There exists a positive constant C such that the super-solutions $\overline{u}_1$ and $\overline{u}_2$ of (2.2) satisfy

$$\overline{u}_1\overline{u}_2 \le C \qquad for \quad t \in (0,\infty).$$

Step 4. There exist positive constants $K_i > 0$, $i = 1, 2$ such that the super-solutions $\overline{u}_1$ and $\overline{u}_2$ of (2.2) satisfy, for $t \in (0,\infty)$,

$$\overline{u}_1 \le K_1 \quad \text{and} \quad \overline{u}_2 \le K_2.$$

Step 5. $(\overline{u}_i - \underline{u}_i) \to 0$ when $t \to \infty$, $i = 1, 2,$

The proofs of Step 1, Step 3, Step 4 and Step 5 are obtained in the same way as the previous case, so we omit the details. To obtain Step 5, we proceed as in Lemma 1.4 in Tello and Winkler [25]. Finally, Theorem 2.2 is a direct consequence of these properties of the solutions. □

3. Comparison principle and asymptotic behavior of solutions

The aim of this section is to relate the solutions of the system of PDE's (1.1)–(1.3) with the solutions of the ODE's system (2.2) that we have studied in the previous section. Under some order relation between initial conditions of both PDE's and ODE's systems, we prove that such order is preserved. Recall that the functions $(\underline{u}_1, \overline{u}_1, \underline{u}_2, \overline{u}_2)$ converge to $(u_1^*, u_1^*, u_2^*, u_2^*)$ as $t \to \infty$, where $(u_1^*, u_1^*, u_2^*, u_2^*)$ are the constant steady states defined in (2.5) and (2.7), respectively, under different restrictions on the coefficients a_1 and a_2. We bound the solution of (1.1) between $\underline{u}_{1,2}$ (lower bound) and $\overline{u}_{1,2}$ (upper bound) to obtain the same qualitative behavior than $\underline{u}_{1,2}$ and $\overline{u}_{1,2}$. The proof follows the *rectangle method* used in Pao [21] for reaction diffusion systems, see also Negreanu and Tello [18,19] and [20] where the method is applied to parabolic–elliptic systems with chemotactic terms.

Thanks to assumption (1.4), we have positive numbers $(\underline{u}_1^0, \overline{u}_1^0, \underline{u}_2^0, \overline{u}_2^0)$ such that

$$0 < \underline{u}_1^0 \le u_1^0(x) \le \overline{u}_1^0, \tag{3.1}$$

$$0 < \underline{u}_2^0 \le u_2^0(x) \le \overline{u}_2^0 \tag{3.2}$$

for all $x \in \Omega$.

In order to prove our main theorem we define the functions

$$\overline{U}_i(x,t) := u_i(t,x) - \overline{u}_i(t) \qquad \underline{U}_i(x,t) := u_i(t,x) - \underline{u}_i(t), \tag{3.3}$$

$$\overline{V}_i(x,t) := v_i(t,x) - \overline{u}_j(t) \qquad \underline{V}_i(x,t) := v_i(t,x) - \underline{u}_j(t), \tag{3.4}$$

for $i, j = 1, 2$, $i \ne j$ where (u_1, u_2, v_1, v_2) and $(\underline{u}_1, \overline{u}_1, \underline{u}_2, \overline{u}_2)$ are the solutions of (1.1)–(1.3) and (2.2), respectively, and we use from now on the usual notation for the positive and negative part of a function f given by

$$(f)_+ = \begin{cases} f & \text{if } f \ge 0, \\ 0 & \text{in other case,} \end{cases} \qquad (f)_- = (-f)_+.$$

We aim to prove that the positive and negative parts $(\overline{U}_i)_+$,$(\underline{U}_i)_-$ are identically zero, and therefore, the solutions inherit the order $\underline{u}_i < u_i < \overline{u}_i$.

Theorem 3.1. *Let* $(u_1^0, u_2^0) \in (L^\infty(\Omega))^2$. *The solution of* (1.1)–(1.3) *with initial data verifying* (3.1)–(3.2) *is bounded and satisfies*

$$\underline{u}_1(t) \le u_1(t,x) \le \overline{u}_1(t), \quad \underline{u}_1(t) \le v_2(t,x) \le \overline{u}_1(t), \quad (x,t) \in \Omega \times (0,\infty),$$

$$\underline{u}_2(t) \le u_2(t,x) \le \overline{u}_2(t), \quad \underline{u}_2(t) \le v_1(t,x) \le \overline{u}_2(t), \quad (x,t) \in \Omega \times (0,\infty)$$

where $(\underline{u}_i, \overline{u}_i)$ *is the solution of the ODE system* (2.2).

To prove Theorem 3.1, we recall the following result, since the proof is similar to Lemma 3.2 in [17], we omit the details.

Lemma 3.1. *Let* $\Omega \in \mathbb{R}^n$ *be a bounded and open domain in* $\mathbb{R}^n$, $p \in \mathbb{N}$, $p \in (\max[\frac{n}{2}, 1], \infty)$ *and* v_i *be the solution of*

$$\begin{cases} -\Delta v_i + \alpha_i v_i = \beta_i u_j, & x \in \Omega, \\ \dfrac{\partial v_i}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

for $u_i \in L^p(\Omega)$, $i, j = 1, 2$ *and* $i \neq j$. *Then, for any* $q \leq \infty$, *the following inequalities hold:*

$$\|(\underline{V}_i)_+\|_{L^q(\Omega)} \leq C(\Omega, p, q)\|\left(\underline{U}_j\right)_+\|_{L^p(\Omega)}.$$

and

$$\|(\underline{V}_i)_-\|_{L^q(\Omega)} \leq C(\Omega, p, q)\|\left(\underline{U}_j\right)_-\|_{L^p(\Omega)},$$

for $i, j = 1, 2$, $i \neq j$ *and* $\underline{V}_i$, $\underline{U}_i$ *defined in* (3.3).

First, we consider $T_{\max}$ defined as the first $t > 0$ such that

$$\limsup_{t \to T_{\max}} \left(\|u_i\|_{L^\infty(\Omega)} + \|v_i\|_{L^\infty(\Omega)} + t\right) = \infty.$$

Then, for any $T < T_{\max}$ we have that the functions (u_1, u_2, v_1, v_2) are continuous and differentiable in $\Omega \times (0, T)$ we can find $c_1(T) \geq 0$ such that

$$\begin{aligned} u_1(x,t) &\leq c_1(T), \qquad u_2(x,t) \leq c_1(T), \\ v_1(x,t) &\leq c_1(T), \qquad v_2(x,t) \leq c_1(T). \end{aligned} \tag{3.5}$$

Lemma 3.2. *Let* $(\overline{u}_2, \underline{u}_2)$ *be the solution of the ODE system* (2.2). *The solutions of* (1.1) *satisfy:*

$$\int_\Omega (\alpha_1 v_1 - \beta_1 \overline{u}_2)_+^2 \leq \int_\Omega (\overline{U}_2)_+^2, \tag{3.6}$$

$$\int_\Omega (\alpha_1 v_1 - \beta_1 \underline{u}_2)_+^2 \leq \int_\Omega (\underline{U}_2)_-^2. \tag{3.7}$$

Proof. Taking the third equation of (1.1) and subtracting on both sides $\overline{u}_2$, we obtain

$$-\Delta v_1 + \alpha_1 v_1 - \beta_1 \overline{u}_2 = \beta_1 \overline{U}_2.$$

If we multiply now by $(\alpha_1 v_1 - \beta_1 \overline{u}_2)_+$ and integrate over Ω, thanks to Young inequality we get

$$\frac{1}{\alpha_1} \int_\Omega |\nabla(\alpha_1 v_1 - \beta_1 \overline{u}_2)_+|^2 + \frac{1}{2} \int_\Omega (\alpha_1 v_1 - \beta_1 \overline{u}_2)_+^2 \leq \frac{1}{2} \int_\Omega (\overline{U}_2)_+^2,$$

which proves (3.6). Since the proof of (3.7) is analogous, we omit the details. □

Proof of Theorem 3.1. We consider $0 < T < \infty$. First we deduce the equation that satisfies $\overline{U}_1$ subtracting the first equations in (1.1) and (2.2), respectively,

$$\begin{aligned} &\frac{\partial \overline{U}_1}{\partial t} - d_1 \Delta \overline{U}_1 + \chi_1 \nabla \overline{U}_1 \nabla v_1 \\ &\leq \beta_1 \left(\chi_1 u_1 u_2 - |\chi_1| \overline{u}_1 \overline{u}_2 + |\chi_1| \overline{u}_1 \underline{u}_2\right) \\ &-\mu_1 a_1 (u_1 u_2 - \overline{u}_1 \underline{u}_2) - \chi_1 \alpha_1 u_1 v_1 + g(u_1) - g(\overline{u}_1), \end{aligned} \tag{3.8}$$

where $g(u_1) := \mu_1 u_1(1-u_1)$. We have

$$u_1u_2 - \overline{u}_1\underline{u}_2 = (u_1-\overline{u}_1)u_2 + (u_2-\underline{u}_2)\overline{u}_1 = \overline{U}_1 u_2 + \underline{U}_2\overline{u}_1.$$

Rewriting some of the above terms in order to simplify, considering two cases, χ_1 positive and negative, we get:

- **Case** $\chi_1 > 0$**.** In (3.8), $|\chi_1| = \chi_1$ and

$$\begin{aligned}
&\beta_1\left(\chi_1 u_1u_2 - |\chi_1|\overline{u}_1\overline{u}_2 + |\chi_1|\overline{u}_1\underline{u}_2\right) - \chi_1\alpha_1 u_1 v_1\\
&= \chi_1\beta_1\left(u_1u_2 - \overline{u}_1\overline{u}_2 + \overline{u}_1\underline{u}_2\right) - \chi_1\alpha_1u_1v_1\\
&= \chi_1\beta_1\left(u_1u_2 - \overline{u}_1\overline{u}_2 + \overline{u}_1\underline{u}_2 - \overline{u}_1u_2 + \overline{u}_1u_2\right) - \chi_1\alpha_1u_1v_1\\
&= \chi_1\beta_1\left[(u_1-\overline{u}_1)u_2 + (u_2-\overline{u}_2)\overline{u}_1 + \overline{u}_1\underline{u}_2\right] - \chi_1\alpha_1u_1v_1\\
&= \chi_1\beta_1\overline{U}_1u_2 + \chi_1\beta_1\overline{U}_2\overline{u}_1 + \chi_1\beta_1\overline{u}_1\underline{u}_2 - \chi_1\alpha_1\left(\overline{U}_1v_1 + \overline{u}_1v_1\right).
\end{aligned} \tag{3.9}$$

Thus, the equation that $\overline{U}_1$ verifies

$$\begin{aligned}
\frac{\partial \overline{U}_1}{\partial t} &= d_1\Delta\overline{U}_1 - \chi_1\nabla\overline{U}_1\nabla v_1 + \overline{U}_1[(\chi_1\beta_1-\mu_1a_1)u_2 - \chi_1\alpha_1v_1]\\
&\quad + \chi_1\beta_1\overline{U}_2\overline{u}_1 - \mu_1a_1\underline{U}_2\overline{u}_1 + \chi_1\overline{u}_1(\beta_1\underline{u}_2 - \alpha_1v_1) + g(u_1) - g(\overline{u}_1).
\end{aligned}$$

We multiply now the previous equation by the test function $(\overline{U}_1)_+$, observing that

$$\overline{U}_1(\overline{U}_1)_+ = (\overline{U}_1)_+^2, \qquad \frac{\partial}{\partial t}\overline{U}_1(\overline{U}_1)_+ = \frac{1}{2}\frac{\partial}{\partial t}(\overline{U}_1)_+^2,$$
$$\int_\Omega \Delta\overline{U}_1(\overline{U}_1)_+ = -\int_\Omega |\nabla(\overline{U}_1)_+|^2.$$

We apply the mean value theorem to g for some

$$\xi_1(x,t) \in (u_1(x,t), \overline{u}_1(t)) \cup (\overline{u}_1(t), u_1(x,t)),$$

i.e., $g(u_1) - g(\overline{u}_1) = g'(\xi_1)\overline{U}_1$, integrating by parts over Ω, the equation for $\overline{U}_1$ becomes

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (\overline{U}_1)_+^2 + d_1\int_\Omega |\nabla(\overline{U}_1)_+|^2\\
&= \underbrace{-\chi_1\int_\Omega (\overline{U}_1)_+\nabla\overline{U}_1\nabla v_1}_{(1)} + \int_\Omega (\overline{U}_1)_+^2 g'(\xi_1)\\
&\quad + \underbrace{\int_\Omega (\overline{U}_1)_+^2(\chi_1\beta_1 - \mu_1a_1)u_2 - \chi_1\alpha_1v_1)}_{(2)} + \underbrace{\chi_1\beta_1\int_\Omega (\overline{U}_1)_+\overline{U}_2\overline{u}_1}_{(3)}\\
&\quad \underbrace{-\mu_1a_1\int_\Omega (\overline{U}_1)_+\underline{U}_2\overline{u}_1}_{(4)} + \underbrace{\int_\Omega (\overline{U}_1)_+\overline{u}_1(\chi_1\beta_1\underline{u}_2 - \chi_1\alpha_1v_1)}_{(5)}.
\end{aligned} \tag{3.10}$$

1. Using integration by parts and the third equation in (1.1), the first term in (3.10) remains

$$-\chi_1\int_\Omega (\overline{U}_1)_+\nabla\overline{U}_1\nabla v_1 = \frac{\chi_1}{2}\int_\Omega (\overline{U}_1)_+^2(\alpha_1v_1 - \beta_1u_2).$$

By relation (3.5) it follows that $|\beta_1 u_2 - \alpha_1 v_1| \leq (\beta_1 + \alpha_1)c_1(T)$ in $\Omega \times (0,T)$ and therefore we can bound the first one

$$-\chi_1 \int_\Omega (\overline{U}_1)_+ \nabla \overline{U}_1 \nabla v_1 \leq \frac{(\beta_1 + \alpha_1)\chi_1 c_1(T)}{2} \int_\Omega (\overline{U}_1)_+^2.$$

2. For the second integral in (3.10), by (3.5), we have

$$|(\chi_1\beta_1 - \mu_1 a_1)u_1 - \chi_1\alpha_1 v_1| \leq (\chi_1(\beta_1 + \alpha_1) + \mu_1 a_1)c_1(T)$$

and we obtain

$$\int_\Omega (\overline{U}_1)_+^2((\chi_1\beta_1 - \mu_1 a_1)u_2 - \chi_1\alpha_1 v_1) \leq c_1(T) \int_\Omega (\overline{U}_1)_+^2.$$

3. For the third integral in (3.10), we use Lemma 2.4 together with Young inequality and we have

$$\begin{aligned}
&\chi_1\beta_1 \int_\Omega (\overline{U}_1)_+ \overline{U}_2 \overline{u}_1 \leq \chi_1\beta_1 \int_\Omega (\overline{U}_1)_+ (\overline{U}_2)_+ \overline{u}_1 \\
&\leq C\chi_1\beta_1 \int_\Omega (\overline{U}_1)_+ (\overline{U}_2)_+ \leq \frac{1}{2} C\chi_1\beta_1 \left[\int_\Omega (\overline{U}_1)_+^2 + \int_\Omega (\overline{U}_2)_+^2 \right].
\end{aligned}$$

4. We repeat the same strategy for the fourth term in (3.10), i.e., applying Lemma 2.4 together with Young inequality, for all $t \in (0,T)$ it results

$$\begin{aligned}
-\mu_1 a_1 \int_\Omega (\overline{U}_1)_+ \underline{U}_2 \overline{u}_1 &\leq \mu_1 a_1 \int_\Omega (\overline{U}_1)_+ (\underline{U}_2)_- \overline{u}_1 \\
&\leq \mu_1 a_1 C \int_\Omega (\overline{U}_1)_+ (\underline{U}_2)_- \\
&\leq \frac{1}{2} \mu_1 a_1 C \left[\int_\Omega (\overline{U}_1)_+^2 + \int_\Omega (\underline{U}_2)_-^2 \right].
\end{aligned}$$

5. For the fifth and last integral in (3.10), we use (3.5) and the bounds obtained in Lemma 3.2, so we find

$$\begin{aligned}
&\int_\Omega (\overline{U}_1)_+ \overline{u}_1 (\chi_1\beta_1 \underline{u}_2 - \chi_1\alpha_1 v_1) \\
&\quad \leq -C\chi_1 \int_\Omega (\overline{U}_1)_+ (\alpha_1 v_1 - \beta_1 \underline{u}_2) \\
&\quad \leq C\chi_1 \int_\Omega (\overline{U}_1)_+ (\alpha_1 v_1 - \beta_1 \underline{u}_2)_- \\
&\quad \leq \frac{1}{2} C\chi_1 \left[\int_\Omega (\overline{U}_1)_+^2 + \int_\Omega (\chi_1\beta_1 \underline{u}_2 - \chi_1\alpha_1 v_1)_+^2 \right] \\
&\quad \leq \frac{1}{2} C\chi_1 \left[\int_\Omega (\overline{U}_1)_+^2 + \int_\Omega (\underline{U}_2)_+^2 \right], \qquad t \in (0,T).
\end{aligned}$$

- **Case $\chi_1 < 0$.** Subtracting the first equations in (1.1) and (2.2), respectively, we get

$$\begin{aligned}\frac{\partial \overline{U}_1}{\partial t} &= d_1\Delta\overline{U}_1 - \chi_1\nabla\overline{U}_1\nabla v_1 + \chi_1\beta_1(u_1u_2 + \overline{u}_1\overline{u}_2 - \overline{u}_1\underline{u}_2) \\ &\quad -\mu_1 a_1(u_1u_2 - \overline{u}_1\underline{u}_2) - \chi_1\alpha_1 u_1 v_1 + g(u_1) - g(\overline{u}_1),\end{aligned} \tag{3.11}$$

where $g(u_1) : \mu_1 = u_1(1-u_1)$. All terms in (3.11) are treated in the same way as in the previous section, i.e., (3.8), except the term corresponding to (3.9), which now is as follows:

$$\chi_1\beta_1\left(u_1u_2 + \overline{u}_1\overline{u}_2 - \overline{u}_1\underline{u}_2\right) - \chi_1\alpha_1 u_1 v_1$$

since

$$\chi_1\beta_1 u_1u_2 - \chi_1\beta_1\overline{u}_1\underline{u}_2 = \chi_1\beta_1[(u_1 - \overline{u}_1)u_2 + \overline{u}_1(u_2 - \underline{u}_2)]$$

and

$$\chi_1\beta_1\overline{u}_1\overline{u}_2 - \chi_1\alpha_1 u_1 v_1 = -\chi_1\alpha_1\left[(u_1 - \overline{u}_1)v_1 + \overline{u}_1\left(v_1 - \frac{\beta_1}{\alpha_1}\overline{u}_2\right)\right].$$

Therefore,

$$\begin{aligned}&\chi_1\beta_1\left(u_1u_2 + \overline{u}_1\overline{u}_2 - \overline{u}_1\underline{u}_2\right) - \chi_1\alpha_1 u_1 v_1 \\ &\quad = \chi_1\beta_1[(u_1 - \overline{u}_1)u_2 + \overline{u}_1(u_2 - \underline{u}_2)] \\ &-\chi_1\alpha_1\left[(u_1 - \overline{u}_1)v_1 + \overline{u}_1\left(v_1 - \frac{\beta_1}{\alpha_1}\overline{u}_2\right)\right].\end{aligned}$$

Multiplying by $(\overline{U}_1)_+$ and integrating over Ω it results

$$\begin{aligned}&\int_\Omega \left[\chi_1\beta_1[\overline{U}_1u_2 + \overline{u}_1\underline{U}_2)] - \chi_1\alpha_1\left[\overline{U}_1 v_1 + \overline{u}_1\left(v_1 - \frac{\beta_1}{\alpha_1}\overline{u}_2\right)\right]\right](\overline{U}_1)_+ \\ &\leq \int_\Omega C[(\overline{U}_1)_+^2 + (\underline{U}_2)_-(\overline{U}_1)_+] - C[(\overline{U}_1)_+^2 + \overline{u}_1\left(v_1 - \frac{\beta_1}{\alpha_1}\overline{u}_2\right)_+(\overline{U}_1)_+] \\ &\qquad \leq C\left[\int_\Omega \overline{U}_1)_+^2 + \int_\Omega(\underline{U}_2)_-^2 + \int_\Omega\left(v_1 - \frac{\beta_1}{\alpha_1}\overline{u}_2\right)_+^2\right] \\ &\qquad\qquad \leq C\left[\int_\Omega \overline{U}_1)_+^2 + \int_\Omega(\underline{U}_2)_-^2\right],\end{aligned}$$

in view of Lemma 3.2. As in the case $\chi_1 > 0$, we obtain similar upper bounds for the enumerate integrals for the commune terms (independent terms of χ_i) in (3.8) and (3.11). After routinary computations, we get the same result if χ_i is negative.

With all the previous calculus, we reach

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\overline{U}_1)_+^2 \leq k_1(T)\left(\int_\Omega(\overline{U}_1)_+^2 + \int_\Omega(\overline{U}_2)_+^2 + \int_\Omega(\underline{U}_2)_-^2\right)$$

for all $t \in (0,T)$, with $k_1(T)$ being a positive constant. In the same way, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\underline{U}_1)_-^2 \leq k_2(T)\left(\int_\Omega (\underline{U}_1)_-^2 + \int_\Omega (\overline{U}_2)_+^2 + \int_\Omega (\underline{U}_2)_-^2\right),$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\overline{U}_2)_+^2 \leq k_3(T)\left(\int_\Omega (\overline{U}_2)_+^2 + \int_\Omega (\overline{U}_1)_+^2 + \int_\Omega (\underline{U}_1)_-^2\right),$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\underline{U}_2)_-^2 k_4(T)\left(\int_\Omega (\underline{U}_2)_-^2 + \int_\Omega (\overline{U}_1)_+^2 + \int_\Omega (\underline{U}_1)_-^2\right).$$

We add the previous inequalities to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (\overline{U}_1)_+^2 + (\underline{U}_1)_-^2 + (\overline{U}_2)_+^2 + (\underline{U}_2)_-^2\right)$$
$$\leq k(T)\left(\int_\Omega (\overline{U}_1)_+^2 + (\underline{U}_1)_-^2 + (\overline{U}_2)_+^2 + (\underline{U}_2)_-^2\right),$$

with $k(T) = \max\{k_i(T) : i = 1...4\}$. Applying Gronwall's lemma and recalling hypothesis (3.1)–(3.2), we have $(\overline{U}_i^0)_+ = (\underline{U}_i^0)_- = 0$, $i = 1,2$ and conclude

$$\int_\Omega \left((\overline{U}_1)_+^2 + (\underline{U}_1)_-^2 + (\overline{U}_2)_+^2 + (\underline{U}_2)_-^2\right) = 0, \qquad \forall t \in (0,T),$$

so, $\overline{U}_{i,+} = \underline{U}_{i,-} = 0$, for $i = 1,2$. Hence we have

$$\underline{u}_1(t) \leq u_1(t,x) \leq \overline{u}_1(t), \quad \underline{u}_2(t) \leq u_2(t,x) \leq \overline{u}_2(t), \quad (x,t) \in \Omega \times (0,T).$$

By Lemma 3.1, we also obtain $\overline{V}_{i,+} = \underline{V}_{i,-} = 0$, for $i = 1,2$ and therefore

$$\underline{u}_1(t) \leq v_2(t,x) \leq \overline{u}_1(t), \qquad \underline{u}_2(t) \leq v_1(t,x) \leq \overline{u}_2(t), \qquad (x,t) \in \Omega \times (0,T).$$

As $T < T_{\max}$ is arbitrary, we take limits as $T \to T_{\max}$ and conclude that $T_{\max} = \infty$ and the proof of the Theorem 3.1 ends. □

4. Existence of the solution and asymptotic behavior

We first consider the local existence of solutions. The result is enclosed in the following lemma.

Lemma 4.1. *Under assumptions (1.5)–(1.6), there exists a unique solution (u_1, u_2, v_1, v_2) to (1.1)–(1.3) in $(0, T_{\max})$ satisfying*

$$u_i, v_i \in C_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_T), \qquad \text{for } i = 1,2 \text{ and any } T < T_{\max}$$

where $T_{\max}$ is a positive number satisfying

$$\limsup_{t \to T_{\max}} \left(\|u_i(t)\|_{L^\infty(\Omega)} + \|v_j(t)\|_{L^\infty(\Omega)} + t\right) = \infty, \qquad i,j = 1,2.$$

Moreover, for $i = 1,2$,

$$u_i(t,x) \geq 0, \quad v_i(x,t) \geq 0, \qquad x \in \Omega, \quad t < T_{\max}.$$

Proof. The proof follows standard fixed point theory, see for instance Horstmann [8], Horstmann and Winkler [9], or Negreanu and Tello [20]. To obtain the regularity, we consider any $T < T_{\max}$, then u_i satisfies

$$u_{it} - \Delta u_i + b_i(t,x) \cdot \nabla u_i + u_i c_i(x,t) = 0, \quad (t,x) \in \Omega_T,$$

where

$$b_i(x,t) = \chi_i \nabla v_i, \quad c_i(x,t) = \chi_i(\beta_i u_j - \alpha_i v_i), \quad \Omega_T = (0,T) \times \Omega$$

for $i,j = 1,2$ and $i \neq j$. Since $u_i \in L^\infty(\Omega_T)$ we have that $v_i \in L^s(0,T : W^{2,q}(\Omega)) \cap L^\infty(\Omega_T)$ for any $s,q < \infty$ and therefore

$$b_i(x,t) \in L^s(0,T : W^{1,q}(\Omega)), \qquad c_i(x,t) \in L^\infty(\Omega_T).$$

Then, thanks to Remark 48.3 (ii) in Quittner–Souplet [22] and following the notation there we have $u_i \in W^{2,1,q}(\Omega_T)$ for any $q < \infty$ which implies that that $u_i \in C^{1+\alpha,\beta}_{x,t}(\Omega_T)$ for any $\alpha, \beta \in (0,1)$. Therefore we have that

$$u_{it} - \Delta u_i \in C^{\alpha,\beta}_{x,t}(\Omega_T) \qquad \text{for any } \alpha,\beta \in (0,1)$$

and we deduce the wished result (see for instance Remark 48.3 (ii) in [22]).

Uniqueness of solutions is obtained by contradiction, following standard arguments. The nonnegativity of u_i is a consequence of the maximum principle. □

Lemma 4.2. *Let v_i be the unique solution of*

$$\begin{cases} -\Delta v_i + \alpha_i v_i = \beta_i u_j \\ \dfrac{\partial v_i}{\partial \vec{n}}|_{\partial\Omega} = 0 \end{cases}$$

for $i = 1,2$, $j = 1,2$, $i \neq j$. Then, it results

$$v_i \geq 0 \tag{4.1}$$

and

$$\|v_i\|_{L^\infty(\Omega)} \leq \frac{\beta_i}{\alpha_i} \|u_j\|_{L^\infty(\Omega)}. \tag{4.2}$$

Proof. The proof of (4.1) is a consequence of Lemma 4.1 and Maximum Principle. To obtain (4.2), we apply again Maximum Principle. □

End of the proof of Theorem 1.1. The global existence of solutions is given by Lemma 4.1, Theorem 3.1 and Lemmas 2.1 and 2.4.

The proof of the asymptotic behavior of the solutions is a consequence of Theorem 3.1, Lemma 4.1 and Theorem 2.1 or Theorem 2.2 for case $a_2 > 1$ and case $a_2 < 1$, respectively. □

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