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On the stability of homogeneous steady states of a chemotaxis system with logistic growth term

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ABSTRACT

We consider a nonlinear PDEs system of Parabolic–Elliptic type with chemotactic terms. The system models the movement of a population "n" towards a higher concentration of a chemical "c" in a bounded domain Ω.

We consider constant chemotactic sensitivity χ and an elliptic equation to describe the distribution of the chemical

$$n_t - d_n \Delta n = -\chi \mathrm{div}(n\nabla c) + \mu n(1-n),$$
$$-d_c \Delta c + c = h(n)$$

for a monotone increasing and Lipschitz function h.

We study the asymptotic behavior of solutions under the assumption of $2\chi|h'| < \mu$. As a result of the asymptotic stability we obtain the uniqueness of the strictly positive steady states.

1. Introduction

Mathematical models of chemotaxis were introduced by Keller and Segel [1] in order to model the movement and aggregation of amoebae responding to a chemical stimulus. In the last four decades, chemotactic terms have been used to model different types of biological phenomena, as angiogenesis, morphogenesis, immune system response, etc. The specific model we are studying features a coupled system of two PDEs: a parabolic equation with a logistic growth term modeling the density of a population n,

$$n_t - d_n \Delta n = -\chi \mathrm{div}(n\nabla c) + \mu n(1-n) \quad \text{in } x \in \Omega,\ t \in (0,T), \tag{1.1}$$

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where d_n, χ and μ are positive constants and $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$. An elliptic equation describes the concentration of a chemical substance c, which acts as the chemoattractant:

$$-d_c \Delta c = h(n) - c \quad \text{in } x \in \Omega, \tag{1.2}$$

with initial datum and boundary conditions

$$\frac{\partial n}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega,\ t \in (0,T), \tag{1.3}$$

$$n(0) = n_0 \quad \text{on } x \in \Omega, \tag{1.4}$$

$$\frac{\partial c}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega. \tag{1.5}$$

Solutions to (1.1)–(1.5) which are biologically meaningful must satisfy

$$n \geq 0, \qquad c \geq 0. \tag{1.6}$$

The function h represents the production of the chemical substance by the living organisms, which, depending on the process, can take different forms. In the literature the function h has different representations:

- $h(n) = n$, see Jäger and Luckhaus [2] and Tello and Winkler [3]
- $$h(n) = s\frac{n}{\beta + n} \tag{1.7}$$
  where c satisfies a parabolic equation (see Orme and Chaplain [4])
- A polynomial function $h(n) = n^p$.

Myerscough et al. [5] study numerically the steady states of (1.1)–(1.5) and (1.7) focusing on the role of boundary conditions. In [5] the authors found non-constant steady states for a range of boundary conditions including (1.3) and (1.5). The parameters studied in [5] are not considered in Theorem 1.2.

We will study the problem (1.1)–(1.5), for a general function h satisfying

$$h \text{ is locally Lipschitz function}, \tag{1.8}$$

there exists a positive constant $\alpha > 0$ such that

$$0 \leq h' \leq \frac{\mu d_c}{2\chi}(1-\alpha). \tag{1.9}$$

We also assume that the initial data $u_0 \in W^{2,p}(\Omega)$ for some $p > N$,

$$\frac{\partial u_0}{\partial \vec{n}} = 0 \quad \text{in } \partial\Omega$$

and there exist positive constants $\underline{n}_0$ and $\overline{n}_0$ such that

$$0 < \underline{n}_0 \leq n_0 \leq \overline{n}_0 < \infty. \tag{1.10}$$

In Section 2 we prove the following theorem.

Theorem 1.1. *Under assumptions (1.8)–(1.10), there exists a unique solution (n, c) to (1.1)–(1.5) and it satisfies*

$$|n-1|_{L^\infty(\Omega)} + |c - h(1)|_{L^\infty(\Omega)} \longrightarrow 0 \quad \textit{when } t \longrightarrow \infty. \tag{1.11}$$

In Section 3 we shall use Theorem 1.1 to study the steady states of (1.1)–(1.5). The result is enclosed in the following theorem:

Theorem 1.2. *Under assumptions (1.8) and (1.9), the strictly positive steady states of problem (1.1)–(1.5) satisfying (1.6) and (1.7) are*

$$n = 1, \quad c = h(1).$$

2. Proof of Theorem 1.1

Expanding the chemotaxis term in (1.1) we obtain

$$n_t - d_n \Delta n = -\chi \nabla n \cdot \nabla c - \chi n \Delta c + \mu n(1-n) \qquad \text{in } x \in \Omega,\ t \in (0,T).$$

Thanks to (1.2), (1.1)–(1.5) becomes

$$n_t - d_n \Delta n = -\chi \nabla n \cdot \nabla c + \frac{\chi}{d_c} n(h(n) - c) + \mu n(1-n) \quad (x,t) \in \Omega \times (0,T), \tag{2.12}$$

$$-d_c \Delta c = h(n) - c \qquad \text{in } x \in \Omega, \tag{2.13}$$

$$\frac{\partial n}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega,\ t \in (0,T), \qquad n(0) = n_0 \quad \text{on } x \in \Omega, \tag{2.14}$$

$$\frac{\partial c}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega. \tag{2.15}$$

We introduce the system of ODEs:

$$\overline{n}_t = \frac{\chi}{d_c} \overline{n} \left(h(\overline{n}) - h(\underline{n}) + \frac{\mu d_c}{\chi}(1 - \overline{n}) \right), \qquad \overline{n}(0) = \overline{n}_0 := \max_{x\in\Omega}\{n_0(x)\}, \tag{2.16}$$

$$\underline{n}_t = \frac{\chi}{d_c} \underline{n} \left(h(\underline{n}) - h(\overline{n}) + \frac{\mu d_c}{\chi}(1 - \underline{n}) \right), \qquad \underline{n}(0) = \underline{n}_0 := \min_{x\in\Omega}\{n_0(x)\}. \tag{2.17}$$

Lemma 2.1. *Under assumptions (1.8)–(1.10), the solutions to (2.16), (2.17) satisfy*

$$0 < \underline{n} \le \overline{n}, \quad \textit{for } t > 0$$

if $\underline{n}_0 \le \overline{n}_0$.

Proof. Since h is a locally Lipschitz function, using (1.8) and (1.9), we know that there exists a unique solution $(\underline{n}, \overline{n})$ in $(-\epsilon, \infty)$ for some $\epsilon > 0$. Since the solution to (2.17) with initial datum $\underline{n}_0 = 0$ is $\underline{n} = 0$, and $\underline{n}_0 > 0$, by uniqueness of the problem (2.16)–(2.17) we obtain $0 < \underline{n}$ for all $t < \infty$. Let $\tilde{n}$ be the solution to the problem

$$\tilde{n}_t = \mu \tilde{n}(1 - \tilde{n}). \tag{2.18}$$

Then, taking initial data $\underline{n}_0 = \overline{n}_0 > 0$, it results in the fact that $\underline{n} = \overline{n} = \tilde{n}$ is the unique solution to (2.16) and (2.17). Uniqueness of the problem (2.18) and the inequality $\underline{n}_0 \le \overline{n}_0$ prove that $\underline{n} \le \overline{n}$ and the proof ends. □

Remark 2.1. Note that, as a consequence of the previous lemma and (1.9), $\overline{n}$ satisfies $\overline{n}_t \ge \mu \overline{n}(1 - \overline{n})$ and $\overline{n}_t \le \mu \overline{n}(1 - \frac{1}{2}\overline{n})$, and then

$$\min\{\overline{u}_0, 1\} \le \overline{u} \le \max\{\overline{u}_0, 2\}. \tag{2.19}$$

Lemma 2.2. *Under assumptions (1.8), (1.9), the solutions to (2.16), (2.17) satisfy*

$$\lim_{t\to\infty} \underline{n} = \lim_{t\to\infty} \overline{n} = 1.$$

Proof. Multiplying by $\frac{1}{\overline{n}}$ in (2.16) and by $\frac{1}{\underline{n}}$ in (2.17), we obtain

$$\frac{\overline{n}_t}{\overline{n}} = \frac{\chi}{d_c}\left(h(\overline{n}) - h(\underline{n}) + \frac{\mu d_c}{\chi}(1-\overline{n})\right), \tag{2.20}$$

$$\frac{\underline{n}_t}{\underline{n}} = \frac{\chi}{d_c}\left(h(\underline{n}) - h(\overline{n}) + \frac{\mu d_c}{\chi}(1-\underline{n})\right). \tag{2.21}$$

Subtracting the above expressions we get

$$\begin{aligned} \frac{d}{dt}\left(Ln\frac{\overline{n}}{\underline{n}}\right) &= \frac{\chi}{d_c}\left[2(h(\overline{u}) - h(\underline{n})) + \frac{\mu d_c}{\chi}(\underline{n}-\overline{n})\right], \\ &\leq \frac{\chi}{d_c}\left[2\max\{h'\}(\overline{n}-\underline{n}) + \frac{\mu d_c}{\chi}(\underline{n}-\overline{n})\right], \end{aligned}$$

and by (1.9), we have the result

$$\frac{d}{dt}\left(Ln\frac{\overline{n}}{\underline{n}}\right) \leq -\alpha(\overline{n}-\underline{n}). \tag{2.22}$$

Integrating (2.22), and by Lemma 2.1 we obtain $Ln\frac{\overline{n}}{\underline{n}} \leq Ln\frac{\overline{n}_0}{\underline{n}_0}$ and then $Ln\underline{n} \geq Ln\overline{n} - Ln\frac{\overline{n}_0}{\underline{n}_0}$ which implies $\underline{u} \geq \overline{n}\frac{\underline{n}_0}{\overline{n}_0}$ and by (2.19) we get

$$\underline{n} \geq \min\{\overline{n}_0, 1\}\frac{\underline{n}_0}{\overline{n}_0} = \alpha_0 > 0. \tag{2.23}$$

Thanks to Mean Value Theorem and integrating (2.22), we get

$$Ln\frac{\overline{n}}{\underline{n}} \leq e^{-\alpha\alpha_0 t}Ln\frac{\overline{n}_0}{\underline{n}_0}.$$

Taking the limit $t \to \infty$ we obtain

$$\lim_{t\to\infty} Ln\frac{\overline{n}}{\underline{n}} = 0. \tag{2.24}$$

By (2.23), (2.19) and Lemma 2.1, thanks to (2.16) and from (2.24) we get

$$\lim_{t\to\infty} \underline{n} = \lim_{t\to\infty} \overline{n} = 1. \quad \square$$

Let $p \in (N, \infty)$ and G be the subset of functions defined by

$$G := \{n \in L^p(0, T : C^0(\overline{\Omega})), \text{ such that } \underline{n} \leq n \leq \overline{n}\},$$

and let $J : G \to L^p(0, T : C^0(\overline{\Omega}))$ be defined by $J(\tilde{n}) = n$, where n is the solution to (2.12), (2.14) for c defined as the solution of the equation

$$-d_c \Delta c = h(\tilde{n}) - c, \quad \text{in } x \in \Omega, \tag{2.25}$$

and (2.15).

Lemma 2.3. *J has a fixed point in G.*

Proof. Applying a maximum principle to (2.13), we obtain

$$\min h(\tilde{n}) \leq \min_{x\in\Omega}\{c(x,t)\} \leq c \leq \max_{x\in\Omega}\{c(x,t)\} \leq \max h(\tilde{n}),$$

and by (1.9) we have

$$\max_{x\in\Omega}\{c(x,t)\} \leq h(\overline{n}), \tag{2.26}$$

$$\min_{x\in\Omega}\{c(x,t)\} \geq h(\underline{n}). \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.12) we get

$$\frac{\chi}{d_c} n(h(n) - h(\overline{n})) \le n_t - d_n \Delta n + \chi \nabla n \cdot \nabla c + \mu n(1-n) \le \frac{\chi}{d_c} n(h(n) - h(\underline{n})).$$

By construction, we can see that $\overline{n}$ is a super solution, and $\underline{n}$ is a lower solution, which implies $\underline{n} < n < \overline{n}$. Since $h(\tilde{n}) \in L^\infty((0,T) \times \Omega)$ then, $\|c\|_{W^{2,q}(\Omega)} \le k(\overline{u}, p)$ for any $q < \infty$. Substituting in (2.12) we obtain that $J(\tilde{u})$ is uniformly bounded in $L^p(0,T : W^{2,p}(\Omega)) \cap W^{1,p}(0,T : L^p(\Omega))$ for some $p > N$ (see for instance Quittner–Souplet [6], Remark 48.3, p. 439). Since $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ is a compact embedding we obtain that $J(G)$ is a relatively compact set of $L^p(0,T : C^0(\overline{\Omega}))$. Applying the Schauder fixed point theorem we obtain the desired result. □

The fixed point of J is a solution to the problem (2.12)–(2.15), and thanks to (1.9) we have uniqueness of solutions. The existence of solutions is obtained for arbitrary $T < \infty$. Taking limits, we obtain the existence of solutions in $(0,\infty)$ satisfying $\underline{n} \le n \le \overline{n}$.

From (2.26), (2.27), we have

$$h(\underline{n}) \le \min_{x\in\Omega}\{c(x,t)\} \le c \le \max_{x\in\Omega}\{c(x,t)\} \le h(\overline{n}), \tag{2.28}$$

and taking limits when $t \to \infty$ we obtain (1.11).

Corollary 2.1. *There exists a unique steady state (n,c) of (1.1)–(1.5) satisfying $n > 0$, and it is $n = 1$ and $c = h(1)$.*

Remark 2.2. In the linear case, $h(n) = n$, assumption (1.9) is satisfied when $2\chi < \mu d_c$.

3. Steady states

We now use Theorem 1.1 to study the steady states of the system (1.1)–(1.5), (1.7). We consider two cases.

Case 1: $\mu = 0$ **and** $s\chi < 4\gamma$. We are looking for the steady states (n,c) to the problem

$$\begin{cases} -\Delta n = -\mathrm{div}(\chi n \nabla c) & \text{in } x \in \Omega \\ -\Delta c = s\dfrac{n}{1+n} - \gamma c & \text{in } x \in \Omega \\ \dfrac{\partial n}{\partial \mathbf{n}} = \dfrac{\partial c}{\partial \mathbf{n}} = 0 & \text{on } x \in \partial\Omega. \end{cases} \tag{3.29}$$

We introduce the unknown $u = ne^{\chi c}$, and substituting in (3.29) u satisfies

$$\begin{cases} -\mathrm{div}(e^{\chi c}\nabla u) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

Since $c \ge 0$, we have $u = constant$. Substituting in (3.29) results in the equations

$$\begin{cases} -\Delta c + \gamma c = s\dfrac{ue^{\chi c}}{1+ue^{\chi c}} & \text{in } \Omega \\ \dfrac{\partial c}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

Taking $-\Delta c$ as a test function in (3.31) we obtain

$$\int_\Omega (\Delta c)^2 d\sigma + \int_\Omega \left(\gamma - s\chi \frac{ue^{\chi c}}{(1+ue^{\chi c})^2}\right) |\nabla c|^2 d\sigma \le 0.$$

Since $\frac{ue^{\chi c}}{(1+ue^{\chi c})^2} \leq 1/4$ we have

$$\int_\Omega (\Delta c)^2 d\sigma + \left(\gamma - \frac{s\chi}{4}\right) \int_\Omega (\nabla c)^2 d\sigma \leq 0. \tag{3.32}$$

If $\frac{s\chi}{4} < \gamma$ we obtain that c is constant and also n. By mass conservation principle, we obtain

$$n = \frac{1}{|\Omega|} \int_\Omega n_0, \qquad c = s \frac{\frac{1}{|\Omega|} \int_\Omega n_0}{(1 + \frac{1}{|\Omega|} \int_\Omega n_0)\gamma}. \quad \square$$

Case 2: $\mu > 0$ and $n \geq \underline{n} > 0$ for some constant $\underline{n}$.

Let $\Omega \subset \mathbb{R}^N$ for $N \geq 1$, then

$$\begin{cases} -\Delta n = -\nabla \chi(n \nabla c) + \mu n(1-n) & \text{in } x \in \Omega \\ -\Delta c = s\dfrac{n}{1+n} - c & \text{in } x \in \Omega, \\ \dfrac{\partial n}{\partial \mathbf{n}} = \dfrac{\partial c}{\partial \mathbf{n}} = 0 & \text{in } x \in \partial\Omega. \end{cases} \tag{3.33}$$

We study the steady states of the above system under the assumption:

$$\text{there exists a constant } \underline{n} > 0 \text{ such that } n \geq \underline{n} > 0 \text{ in } \Omega. \tag{3.34}$$

Notice that $|h'(n)| \leq s$ for $n \geq 0$ then, thanks to Theorem 1.1 we have that under the assumptions

$$d_c = 1, \qquad 2\chi < \mu,$$

the positive steady states satisfying (3.34) are given by

$$n = 1, \qquad c = \frac{s}{2\gamma}.$$

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